The Logical Analysis of Plurals and Mass Terms: A Lattice-theoretical Approach

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1 Introduction

The weekly Magazine of the German newspaper Frankfurter Allgemeine Zeitung regularly issues Marcel Proust’s famous questionnaire which is answered each time by a different personality of West German public life. One of those recently questioned was Rudolf Augstein, editor of Der Spiegel; his reply to the question: “Which property do you appreciate most with your friends?” was

(1) “that they are few.”

Clearly, this is not a property of any one of Augstein’s friends; yet, even apart from the esprit it was designed to display the answer has a straightforward interpretation. The phrase (1) predicates something collectively of a group of objects, here: Augstein’s friends.

As it is well known, collective predication is a rather pervasive phenomenon in natural language, as the following sample of sentences shows:

(2) The children built the raft.
(3) The Romans built the bridge.
(4) Tom, Dick, and Harry carried the piano upstairs.
(5) The playing cards are scattered all over the floor.
(6) The members of the committee will come together today.
(7) Mary and Sue are room-mates.
(8) The girls hated each other.

There is a striking similarity between collective predication and predication involving mass nouns.

(9) (a) The children gather around their teacher.
(b) The water gathers in big pools.
Moreover, a characteristic feature of mass terms, their *cumulative reference property*, can be imitated by plurals.

(10) (a) If \( a \) is water and \( b \) is water then the sum of \( a \) and \( b \) is water.
    (b) If the animals in this camp are horses, and the animals in that camp are horses, then the animals in both camps are horses.

All this has been observed and discussed in the literature although the noted parallelism has perhaps not been stressed too much. As it can be seen from Pelletier's 1979 volume, however, there is much disagreement about the proper way of attacking the logical problems posed by plurals and mass terms. From a semantic point of view the basic question is: what do mass terms and plural expressions denote? Some have thought that in order to be able to give a satisfactory answer to this question it is necessary to give up or at last extend the underlying set theory and to define new kinds of objects, for instance *ensembles* (Bunt 1979) or *Kollektionen* (Blau 1979). I think, however, that we can retain the usual set-theoretic metalanguage and simply enrich the structure of our models as to account for properties like cumulative reference. On my view, such properties are also not secured by *defining* some plural or mass term denotations out of others through set-theoretic manipulations; they all should be recognized as simply being there. What we rather should try to discover, then, is the network of the various relations which they enter and through which they are tied together. In the case of group and mass objects this picture naturally leads to the notion of lattice structure, an idea which is, again, not new: it is inherent in mereological predicate logic and the Calculus of Individuals as developed by Leonard and Goodman (1940) and Goodman and Quine (1947). However, its possible use in the present context has perhaps been obscured by reductionist ontological considerations which are, in my opinion, quite alien to the purpose of logically analyzing the inference structures of natural language. Our guide in ontological matters has to be language itself, it seems to me. So if we have, for instance, two expressions \( a \) and \( b \) that refer to entities occupying the same place at the same time but have different sets of predicates applying to them, then the entities referred to are simply not the same. From this it follows that my ring and the gold making up my ring are different entities; they are, however, connected by what I shall call the *constitution relation*: There is exactly one portion of matter making up my ring at a time. A constitution relation \( C \) has been explicitly introduced into the discussion by Parsons (1979). Sharing his intuitions on this point I shall provide a similar 2-place relation, \( \rightarrow \), for *constitutes* or *makes up*. Its semantic counterpart in the theory to be presented below, the "materialization" function \( h \), lies at the heart of my reconstruction of the ontology of plurals and mass terms: individuals are created by linguistic expressions involving different structures even if the portion of matter making them up is the same. Consider the example from Blau (1979) (imagine that there is a deck of playing cards on the table):

(11) (a) the cards
    (b) the deck of cards

While the portions of matter denoted by (11a) and (11b) are the same, I consider the individuals as being distinct. (11a) refers to the pure collection of objects, and in many
contexts (11b), too, refers just to this collection. In general, however, the introduction of a collective term like (11b) is indicative of connotations being added enough for it to refer to a different individual; for instance, a committee is not just the collection of its members, etc. Note, by the way, that the transition to an intension function would be of no help here. There might be two different committees which necessarily consist of exactly the same members.\(^7\)

It might be thought, then, that collective predication is just the context in which pairs of expressions like (11a, b) refer to collections and thus are coreferential. This is not so, however, as can be seen from the following example. Imagine that there are several decks of cards, a blue one, a green one, etc., lying on a table.\(^8\) Then the two following sentences do not mean the same although number consecutively is a collective predicate (the German word is durchnumerieren).

(12) (a) The cards on the table are numbered consecutively.
(b) The decks of cards on the table are numbered consecutively.

By contrast, spatio-temporal collective predicates do refer to the pure collection, or, as I conceive it, the portion of matter making up the individual in question. Examples are be-on-the-table, occupy, etc. I call such predicates invariant. So the following (a) sentences are indeed equivalent to their corresponding (b) sentences.

(13) (a) The cards are on the table.
(b) The decks of cards are on the table.
(14) (a) The stars that presently make up the Pleiades galactic cluster occupy an area that measures 700 cubic light years.\(^9\)
(b) The Pleiades galactic cluster occupies an area etc.

In the following I shall distinguish between pure plural individuals involved in (12) and collections in the portions-of-matter sense referred to in (13) and (14). The former I call (individual) sums or plural objects; they respect levels of “linguistic comprehension” as shown by (12). By contrast, collections do not, they typically merge those levels. Sums and collections are similar, however, in that they both are just individuals, as concrete as the individuals which serve to define them, and of the same logical type as these. The latter feature is important because there is no systematic type ambiguity inherent in predicates like carry, build, demolish, defend, etc. As to the question of concreteness of sums of concrete objects, I agree with the intuitions of those who say that an aggregate of objects like a heap of playing cards which can be shuffled, burned, etc., is simply not an abstract entity like a set (see Burge 1977; Blau 1979). What is more important, however, is the fact that the set approach to plural objects\(^10\) does not carry over to the case of mass terms, thus missing the structural analogy between the two cases. Inherent in the notions of a set is atomicity which is not present in the linguistic behaviour of mass terms.

Before I go on to present my own approach I want to say something about what ter Meulen calls nominal mass terms. Typical examples are stuff names like gold in sentences like gold has the atomic number 79. But there is also the time-honored sentence water is widespread in which the term water has apparently a somewhat different status. It seems
to refer to the concrete “scattered individual” that you just find everywhere, hence Quine’s analysis in terms of mereology. In this sense the sentence should be synonymous to the water (on earth) is widespread. Of the same type is the use of gold in America’s gold is stored in Fort Knox. Here, again, a concrete object is referred to by America’s gold, namely the material fusion of all quantities of the US gold reserve. So there can be no doubt that some notion of fusion is needed to account for definite descriptions involving predicative mass terms (the μ-operator defined below does just this). Genuine stuff names, however, are something else. Substances are abstract entities and cannot be defined in terms of their concrete manifestations. The question, then, is of what kind the connections are that are intuitively felt between substances and their quantities. Take water, for instance. A quantity is water if it displays the internal structure of water, that is H₂O. But this relation is not a logical one. Or else we might look for substance properties which carry over to the quantities of the substance in question. Water is a liquid and yet, all concrete water might be frozen. So we have to go over to dispositional properties, getting more and more involved into our knowledge of the physical world ... What I am getting at is that nominal mass terms do not seem to have a proper logic. Be this as it may, this issue is completely independent of the lattice structure that governs the behaviour of predicative mass terms and plural expressions; it is only this structure that I want to address myself to in the present paper.11

Now what I am going to propose, then, is basically the following. First of all, let us take seriously the morphological change in pluralization, which is present in many natural languages, and introduce an operator, “∗”, working on 1-place predicates P, which generates all the individual sums of members of the extensions of P. Such a starred predicate now has the same cumulative reference property as a mass predicate, it is closed under sum formation: any sum of parts which are ∗P is again ∗P. This property gives rise to the introduction of a Boolean structure on the domain of discourse, E: technically, E becomes an atomic Boolean algebra which is to be taken complete so that every subset of E possesses a sum. Now let ||·|| be the denotation function in a model, and ||P|| the extension of P. Then ||∗P||, the extension of ∗P, can be defined in terms of ||P|| as the complete join-subsemilattice12 in E generated by ||P||. This construction is the mathematical expression of the closure property referred to above. The set A of atoms of E consists of the “singular objects”, like this card, that deck of cards, etc. Among them are all the different portions of matter, like the water making up this ice cube.

Now, let a and b denote two atoms in A. Then there are two more individuals to be called below a + b and a ⊕ b, a + b is still a singular object in A, the material fusion of a and b; a ⊕ b is the individual sum or plural object of a and b. The theory is such that a + b constitutes, but is not identical with, a ⊕ b. This looks like a wild platonistic caprice strongly calling for Occam’s Razor. Language, however, seems to function that way. Take for a, b two rings recently made out of some old Egyptian gold. Then the rings, a ⊕ b, are new, the stuff, a + b, is old.

Sums are partially ordered through the intrinsic ordering relation “≤”, on E which is expressed in the object language by the 2-place predicate “Π”. It is called the individual part relation (i-part relation, for short) and satisfies the biconditional

\[
(15) \quad a \Pi b \leftrightarrow a + b = b.
\]
The semantic counterpart of (15) is the well-known Boolean relation (where \(\sqcup\) denotes the join operation on \(E\)):

\[
(16) \quad \|a\| \leq \|b\| \quad \text{iff} \quad \|a\| \sqcup \|b\| = \|b\|
\]

Now if \(P\) is a 1-place predicate and \(*P\) the corresponding plural predicate let \(\circ P\) (the proper plural predicate of \(P\)) be true of exactly the non-atomic sums in the extension of \(*P\). Then we can define the sum and the proper sum of the \(P\)'s, \(\sigma xP_x\) and \(\sigma^*xP_x\), respectively, as the supremum of all objects that are \(*P\) and \(\circ P\), respectively ("\(\cup\)" is the description operator):

\[
(17) \quad \begin{align*}
(\alpha) & \sigma xP_x := \omega(\star P_x \land \forall y(\star P_y \rightarrow y\Pi x)). \\
(\beta) & \sigma^*xP_x := \omega(\circ P_x \land \forall y(\star P_y \rightarrow y\Pi x)).
\end{align*}
\]

\(\sigma^*xP_x\) carries the presupposition that there are at least two \(P\)'s; in this case \(\sigma xP_x\) and \(\sigma^*xP_x\) coincide.

The material fusion of the \(P\)'s, denoted by \(\mu xP_x\), becomes that object which constitutes the sum of the \(P\)'s:

\[
(18) \quad \mu xP_x := \omega(x \triangleright \sigma xP_x)
\]

The \(\mu\)-operator has the effect of merging levels of comprehension, as I called it above. Thus we will have \(\mu xP_x = \mu xQ_x\), if \(P\) stands for is a card of one of the card decks, and \(Q\) for is a card deck. \(\sigma xP_x\) and \(\sigma xQ_x\) are not the same, however. If we have two decks of Bridge cards, for instance, \(\sigma xP_x\) contains 104 atoms whereas \(\sigma xQ_x\) contains only 2 atoms.

There is a second ordering relation, called the material part (m-part) relation and denoted by "\(\top\)". It establishes a partial order on portions of matter, but only a preorder, called \(\leq_m\), on the whole domain of individuals. Objects which are m-parts of one another are materially equivalent in that they have the same portion of matter constituting them. If \(a\) is an i-part of \(b\) then \(a\) is an m-part of \(b\); symbolically:

\[
(19) \quad a \sqcap b \rightarrow a \sqcup b
\]

In order to explain the meaning of "\(\top\)" more precisely let me supply the remaining concepts of the model structure to be defined below. In addition to the domain of individuals, \(E\), there is a set \(D\) which is endowed with a join operation "\(\sqcup\)" making \(D\) into a complete, but not necessarily atomic, join-semilattice. \(D\) is partially ordered by its intrinsic ordering relation, \(\leq\), defined by

\[
(20) \quad x \leq y \quad \text{iff} \quad x \sqcup y = y
\]

\(D\) represents the set of all individual portions of matter in the model, and as such, it is considered as a subset of \(\mathcal{A}\), the set of atoms of \(E\). The relation (20) stands for a material part-whole relation.\(^{15}\) It can be used to order the individuals of \(E\) materially. For this I postulate for every model the existence of a semilattice homomorphism \(h\) from \(E \setminus \{0\}\)
to $D$ such that $h$ is the identity function on $D$ and commutes with the supremum
operator. Then $h$ is order preserving, i.e. we have

$$x \leq y \Rightarrow h(x) \leq h(y)$$

for all $x, y \in E$ with $x \neq 0$. I can now define the relation $\leq_m$ mentioned above by

$$x \leq_m y \iff h(x) \leq h(y) \quad (x, y \in E \setminus \{0\}).$$

$\leq_m$ is the semantic counterpart of "$\supset$". Obviously, $\leq_i$ is coarser than $\leq_m$, a fact that is
syntactically expressed by (19). In its intended interpretation, then, which is formally
implemented into the model structure, $a \supset b$ can be read as follows: "the portion of
matter constituting $a$ is $m$-part of the portion of matter constituting $b".  

One-place predicates are interpreted as subsets of $E \setminus \{0\}$. (Here, the null element $0$
is assigned the role of a dummy object to which no predicate applies and which can take
care of denotation gaps arising in our description theory; for details see Link (1979,
5.2).) If $P$ is a mass term the extension $\|P\|$ of $P$ in a given model should be a set of
portions of matter which is closed under joins. So $\|P\|$ becomes a subsemilattice of $D$.
This completes the informal description of the Boolean model structure $\mathcal{B} = \langle E, A, D, h \rangle$
and of the LPM-model $M = \langle \mathcal{B}, \| \cdot \| \rangle$. For precise definitions see below.

If $P$ is a predicate we can introduce its mass term correspondent, called "$^m P"; if $\|P\|$ is
the extension of $P$ in a model, define

$$\|^m P\| := \{ x \in D | x \leq \sup h(\|P\|) \}$$

If $P$ is already a predicative mass term we have, by this definition, $\|P\| \subseteq \|^m P\|$. Mass
term correspondents occur in natural language.\(^{14}\) Consider

(24) \hspace{1cm} (a) There is apple in the salad.
\hspace{1cm} (b) $\forall x (^m P x \land Q x)$

(24 b) is a formalization of (24 a), with $P$ for is an apple, $^m P$ for is apple, and $Q$ for is in the
salad. From this we can infer, intuitively as well as formally:

(25) \hspace{1cm} (a) There are apple parts in the salad.
\hspace{1cm} (b) $\forall x \forall y (^m P x \land y \supset x \land Q y)$

We have, in fact, the following theorem (cf. (T. 14) below):

(26) $^m P a \rightarrow \forall y (^m P y \land a \supset y)$

Returning now to the model structure consider once again the extension of a 1-place
predicate $P$. It can contain atoms as well as proper i-sums. This general case of a mixed
extension is exemplified by predicates like carry the piano: think of Tom, Dick, and Harry
(together), Obelix (all by himself), etc. Common nouns and intransitive verbs like die,
however, seem to admit only atoms to their extension. I call such predicates distributive.
(27) \( \text{Distr}(P) \leftrightarrow \forall x(Px \rightarrow \text{At } x) \)

(here the predicate \( \text{At} \) stands for the property of being an atom in the model). To illustrate take the intuitively valid inference from (a) to (b) in (28).

(28) (a) John, Paul, George, and Ringo are pop stars.
(b) Paul is a pop star.

This inference can be formally represented if we consider \( \text{pop star} \) as a distributive predicate \( P \) in the sense of (27). In this case the extension of \( \star P \) is closed under non-zero i-parts (see theorem (T. 10) below), so every atom of an i-sum which is \( \star P \) itself \( \star P \), hence it is a \( P \). In symbolic form the inference (28) looks like this:

(29) (a) \( \star P(a \oplus b \oplus c \oplus d) \)
(b) \( \text{Distr}(P) \)
(c) \( b \Pi a \oplus b \oplus c \oplus d \)
(d) \( \star Pb \)
(e) \( Pb \)

In a similar way we have the following valid inference with the distributive predicate \( Q \) for \( \text{die} \) (\( P \) stands for \( \text{animal} \)):

(30) (a) The animals died. So every animal died.
(b) \( \star \forall x(Px) \Rightarrow \forall x(Px \rightarrow Qx) \)

For non-distributive predicates we have, of course, no such result, witness \( \text{carry: a} \oplus b \) might be in the extension of \( \text{carry} \) while \( a \) or \( b \) alone is not. Notice that these predicates enter formalizations unstarrred. To illustrate,

(31) (a) The children gather around their teacher
(b) \( Q(\sigma xPx) \) (with \( Q \) for \( \text{gather etc.} \)).

I think there is no harm in accepting this systematic difference; i.e. distributive predicates working on plural terms have to be starred, all the other predicates must not be. For distributivity seems to be a lexical feature. If we have a formal translation procedure the predicates have to be subcategorized accordingly.

There is one more operator, \( \text{"} \cap \text{"} \), in the logic LPM to be presented below. \( \cap P \), for a predicate \( P \), is to be read as \( \text{partakes in} \ P \). I introduce this operator in order to be able to distinguish between the plural terms \( \text{the children} \) and all the children. It seems to me that in \( \text{all the children built the raft} \) it is claimed that every child took part in the action whereas in \( \text{the children built the raft} \) it is only said that the children somehow managed to build the raft collectively without presupposing an active role in the action for every single child. This intuition enters the formalization of the two phrases given below. I want to stress, however, that the operator \( \text{"} \cap \text{"} \) can only be partially characterized in view of the essentially pragmatic nature of its intended interpretation. So let me formulate just the following meaning postulates for \( \text{"} \cap \text{"} \) which seem to be plausible:
I conclude the informal part of the paper with some more formalizations of natural language sentences involving plurals and mass terms. Most of the principles governing their logic that have been mentioned in the literature come out valid in the system LPM below. Some of them are instantiated here, like Massey’s plurality principles of symmetry, expansion, and contraction (see Massey (1976)). What I did not treat in LPM, however, are any “downward” closure properties that are somehow felt to be present in the behavior of mass terms (“a part of water is still water”). Such principles can be added when a careful linguistic analysis has succeeded in giving them a form that takes care, in particular, of the problem of minimal parts (is every m-part of lemonade really lemonade again?). For some discussion of downward closure properties see, e.g., Bunt (1979) and Hoepelman and Rohrer (1980).

(33) (a) A child built the raft  
     \(Px: x\) is a child  
     \(Qx: x\) built the raft  

(b) \(\forall x(Px \land Qx)\);  

(34) (a) Children built the raft.  
     \(\forall x(Px \land Qx)\);  

(b) \(\forall x^*(Px \land Qx)\);  

(35) (a) The child built the raft.  
     \(\forall y(y = \alpha Px \land Qy)\);  

(b) \(Px, Qx: d\to\).  

(36) (a) The children built the raft.  
     \(\forall y(y = \sigma^*Px \land Qy)\);  

(b) \(Px, Qx: d\to\).  

(37) (a) Every child saw the raft.  
     \(\forall x(Px \rightarrow Qx)\);  

(b) \(Px: d\to\).  
     \(Qx: x\) saw the raft  

(38) (a) All the children built the raft.  
     \(\forall y(\alpha^*Px \rightarrow Qy)\);  

(b) \(Px, Qx\) as in (33)  

\[Q := \lambda x(Qx \land \alpha xP \rightarrow \neg Qx).\]  

(39) (a) Tom and Dick carried the piano upstairs.  
     \(a: Tom, b: Dick,\)  
     \(Px: x\) carried the piano upstairs  

(b) \(P(a \oplus b);\)  

(40) (a) Tom and Dick carried the piano upstairs, so Dick and Tom carried it upstairs.  
     \(P(a \oplus b) \rightarrow P(b \oplus a)\) (symmetry)  

(b) \(\alpha^*P(a \oplus b) \land Pc \rightarrow \alpha^*P(a \oplus b \oplus c)\) (expansion)  

(41) (a) John and Paul are pop stars and George is a pop star, so John, Paul, and George are pop stars.  
     \(\alpha^*P(a \oplus b) \land Pc \rightarrow \alpha^*P(a \oplus b \oplus c)\) (expansion)  

(b) John, Paul, George, and Ringo are pop stars, so Paul and Ringo are pop stars. (see (28))  

(c) \(\alpha^*P(a \oplus b \oplus c \oplus d) \rightarrow \alpha^*P(b \oplus d)\) (contraction)  

(42) (a) (All) water is wet.  
     \(\forall x(Px \rightarrow Qx);\)  

(b) \(Px: x\) is a quantity of water  

(43) (a) (All) water is wet.  
     \(\forall x(Px \rightarrow Qx);\)  

(b) \(Q(\alpha xPx);\)  

(44) (a) The water of the Rhine is dirty.  
     \(Px: x\) is a quantity of water  

(b) \(Q(\alpha xPx);\)  

(45) (a) \(\alpha \in \mathbb{R}\)  

(b) \(\alpha \in \mathbb{R}\)  

(46) (a) \(\alpha \in \mathbb{R}\)  

(b) \(\alpha \in \mathbb{R}\)
2 The Logic of Plurals and Mass Terms (LPM)

LPM is a first order predicate calculus with the usual logical constants “¬”, “∨”, “∧”, “→”, “↔”, “∀”, “∃”, the description operator denoted by “v”, and the abstraction operator “λ”. The syntactic variables are, for formulas of LPM, “ϕ”, “ψ”, “χ”; for individual terms, “a”, “b”, “c”; for variables, “x”, “y”, “z”; for (1-place) predicates, “P”, “Q”. These symbols can also appear with primes and indices. For metalinguistic (definitional) identity the symbol “≡” (‘:=’) will be used. The set of 1-place predicate constants contains two specified subclasses: the set MT of *predicative mass terms* and the set DP of *distributive predicates*. MT and DP are taken to be disjoint sets.

As special *primitive symbols* of LPM we have a 1-place predicate symbol “F”, three 2-place predicate constants, “Π”, “→”, “↔”, and two operators on 1-place predicates, “*” and “∧”. The intended interpretations are the following. Ea stands for “a exists”; a乙b for “a is an individual part (i-part) of b”; a→b for “a is a material part (m-part) of b”, a↔b for “a constitutes or makes up b”; *P for “the plural predicate of P”; 1P for “partakes in P”.

Now I introduce a number of *defined expressions*.

1 *General first order abbreviations.* Let P, Q be 1-place predicates, R a 2-place predicate, and a an individual term.

(D.1) \[ P \subseteq Q \leftrightarrow \land x (Px \rightarrow Qx) \]
(D.2) \[ P = Q \leftrightarrow P \subseteq Q \land Q \subseteq P \]
(D.3) \[ P \oplus Q := \land x (Px \lor Qx) \]
(D.4) \[ I_a := \land x (x = a) \]

Furthermore, the usual definitions for the following formulas involving R are assumed: Refl(R) ("R is reflexive"), Trans(R) ("R is transitive"), Sym(R) ("R is symmetric"), Antisym(R) ("R is antisymmetric"). We then have, with PrO(R) for "R is a preordering relation", PoR(R) for "R is a partial ordering relation", and Equ(R) for "R is an equivalence relation":

(D.5) \[ PrO(R) \leftrightarrow \text{Refl}(R) \land \text{Trans}(R) \]
(D.6) \[ PoR(R) \leftrightarrow \text{Refl}(R) \land \text{Trans}(R) \land \text{Antisym}(R) \]
(D.7) \[ Equ(R) \leftrightarrow \text{Refl}(R) \land \text{Trans}(R) \land \text{Sym}(R) \]
2 Defined predicate constants. With two individual terms \( a, b \), let \( a = b \) stand for "\( a \) equals \( b \)". \( a \sim b \) for "\( a \) is \( m \)-equivalent to \( b \)". \( \forall a \) for "\( a \) is an atom", and \( Mpa \) for "\( a \) is a portion of matter". Define

\[
\begin{align*}
(D.8) \quad a = b & \iff a \Pi b \land b \Pi a \\
(D.9) \quad a \sim b & \iff a \ominus b \land b \ominus a \\
(D.10) \quad \forall a & \iff \forall x(x \Pi a \to x = a) \\
(D.11) \quad Mpa & \iff a \triangleright a 
\end{align*}
\]

3 Defined predicate operators. Let \( P \) be a 1-place predicate and \( a \) an individual term. The proper plural predicate \( \ast P \) of \( P \) and the mass term correspondent to \( P \), \( \ast P \), are then defined as follows.

\[
\begin{align*}
(D.12) \quad \ast Pa & \iff \ast P a \land \lnot \forall a \\
(D.13) \quad \ast P a & \iff \forall y(\ast P y \land \lnot \exists z(z \triangleright y)).
\end{align*}
\]

4 Defined individual terms. With 1-place predicates \( P \) and individual terms \( a, b \), define the \( i \)-sum of the \( P \)'s, the proper \( i \)-sum of the \( P \)'s, the \( i \)-sum of \( a \) and \( b \), the material fusion of the \( P \)'s, and the material fusion of \( a \) and \( b \), respectively:

\[
\begin{align*}
(D.14) \quad \sigma x P x & := \psi(\ast P x \land \lnot P y \to y \Pi x)) \\
(D.15) \quad \sigma x P x & := \psi(\ast P x \land \lnot P y \to y \Pi x)) \\
(D.16) \quad a \oplus b & := \sigma x(I_a \oplus I_b) x \\
(D.17) \quad \mu x P x & := \psi(x \triangleright \sigma x P x) \\
(D.18) \quad a + b & := \mu x(I_a \oplus I_b) x
\end{align*}
\]

5 Special abbreviative formulas. Let \( P \) be a 1-place predicate.

\[
\begin{align*}
(D.19) \quad \text{Distr}(P) & \iff \forall x(P x \to \forall x) \\
(D.20) \quad M(P) & \iff \forall x(P x \to Mpx) \\
(D.21) \quad \text{Inv}(P) & \iff \forall x(\forall y(\lnot x \sim y \to (P x \leftrightarrow P y))).
\end{align*}
\]

\text{Distr}(P) \text{ is to be read as "\( P \) is distributive" (i.e. \( P \) is true of atomic individuals only),}
\text{\( M(P) \) as "\( P \) is a material predicate" (i.e. it is true of portions of matter only). \( \text{Inv}(P) \)
\text{means: "\( P \) is an invariant predicate" (i.e. it is closed under substitution of m-equivalent terms).}

6 Meaning postulates for "\( \lnot \)".

\[
\begin{align*}
(MP.1) \quad \forall x(\lnot P x & \to \forall y(x \Pi y \to P y)) \\
(MP.2) \quad \text{Distr}(P) & \to \forall x(\lnot P x \leftrightarrow P x)
\end{align*}
\]

i.e. for all \( x \), if \( x \) partakes in \( P \), then \( x \) is an i-part of a \( y \) which is \( P \) (MP.1); if \( P \) is
distributive, then for all \( x \), \( x \) partakes in \( P \) just in case \( x \) is \( P \) (MP.2).
The semantics of LPM. I shall employ the usual set-theoretic terminology (see, for instance, Eberle (1970), Link (1979)), with "⊆" for set inclusion and "f[Y]" for the f-image \( \{ f(x) \mid x \in Y \} \) of \( Y \subseteq X \) under the function \( f : X \to Z \). Moreover, some elementary notions of lattice theory are needed (see, e.g., Grätzer (1971), Sikorski (1969) for their definition) partial ordering relation, partially ordered set ("poset"), 1-element, 0-element, atom, semi-lattice, join-semilattice, lattice, sublattice, ideal, principal ideal, complete lattice, Boolean lattice (algebra), atomic lattice, (semi) lattice homomorphism.

The supremum of a subset \( Y \) of a poset \( X \), relative to its ordering relation \( ≤ \), if it exists, will be denoted by \( \text{sup}_E Y \), the principal ideal generated by an element \( x \in X \), where \( X \) is a lattice, by \( (x) \). So we have \( (x) = \{ y \in X \mid y \leq x \} \). If \( X \) is a complete (semi)lattice, and \( Y \subseteq X \), then \( [Y] \) denotes the complete sub(semi)lattice generated by \( Y \).

I now provide a model-theoretic interpretation of the formal system LPM.

(D.22) A Boolean model structure with homogeneous kernel ("boosk") is a quadruple

\[ \mathfrak{B} := \langle E, A, D, h \rangle \]

such that

1. \( E \) is a complete atomic (c.a.) Boolean algebra, with join operation \( \sqcup \) and the intrinsic ordering relation \( ≤_i \);
2. \( A \subseteq E \) is the set of atoms in \( E \);
3. \( D \subseteq A \) is a complete join-semilattice with join \( \sqcup \) and ordering relation \( ≤_i \);
4. \( h : E \setminus \{ 0 \} \to D \) is a semilattice homomorphism such that

   (i) \( hD = id_D \), i.e. \( h(x) = x \) for all \( x \in D \),

   (ii) \( h(\text{sup}_≤ B) = \text{sup} h[B] \) for all \( B \subseteq E \setminus \{ 0 \} \).

It follows from this definition that we have, in particular, for all \( x, y \in E \setminus \{ 0 \} \):

\[
\begin{align*}
(47) \quad h(x \sqcup_i y) & \equiv h(x) \sqcup h(y) \\
(48) \quad x ≤_i y & \implies h(x) ≤ h(y)
\end{align*}
\]

The homomorphism \( h \) induces a second ordering relation on \( E \setminus \{ 0 \} \) which will be called the material part relation on \( E \setminus \{ 0 \} \) and denoted by \( ≤_m \):

\[
\begin{align*}
(\text{D.23}) \quad x ≤_m y & \iff h(x) ≤ h(y) & (x, y \in E \setminus \{ 0 \})
\end{align*}
\]

\( ≤_m \) is only a preordering relation. It gives rise to an equivalence relation, defined by

\[
\begin{align*}
(\text{D.24}) \quad x \sim_m y & \iff x ≤_m y \text{ and } y ≤_m x & (x, y \in E \setminus \{ 0 \}),
\end{align*}
\]

and thus to a partition of \( E \setminus \{ 0 \} \) into equivalence classes each containing all the individuals that are made up by the same portion of matter.

In the following definition of a model for LPM I am going to interpret only 1-place predicates, for simplicity (except for the special 2-place predicate constants). All the other many-place predicates receive their usual first order interpretation.

(D.25) A model for LPM is an ordered pair \( M = (\mathfrak{B}, || \cdot ||) \) such that

1. \( \mathfrak{B} = \langle E, A, D, h \rangle \) is a boosk (\( E \) is the domain of individuals in \( M \), \( A \) the set of atoms in \( M \), \( D \) the set of portions of matter in \( M \), and \( h \) the "materialization function" in \( M \)).
2 $\| \cdot \|$ is a first order assignment of denotations to the primitive expressions of LPM such that

(i) $\| a \| \in E$ if $a$ is an individual constant;
(ii) $\| P \| \subseteq E \setminus \{ \emptyset \}$ if $P$ is a 1-place predicate constant;
(iii) $\| P \| \subseteq A$ if $P \in DP$;
(iv) $\| P \|$ is a complete subsemilattice of $D$ if $P \in MT$;
(v) $\{ x \in E \mid x \neq 0 \land \exists y \in \| P \| \text{ such that } x \leq y \}$;
(vi) $\| P \| = \| P \|$ if $P \in DP$.

The model is such that predicates are interpreted in the non-zero elements of $E$ (condition (ii)). Conditions (iii), (iv) guarantee the special properties of the distributive and mass predicates, respectively. (v) and (vi) are designed to validate the meaning postulates (MP.1) and (MP.2) above, respectively.

Let $M = (\mathfrak{B}, \| \cdot \|)$ be a model for LPM. The usual first order semantical recursion rules are assumed, where quantification is taken to run over the non-zero elements of the domain of individuals only. The truth and denotation conditions for the special primitive symbols of LPM are defined as follows ("1" stands for "true"; bivalence is assumed):

(D.26) $\| Pa \| \doteq 1$ iff $\| a \| \in \| P \|$
(D.27) $\| \lambda x P x \| \doteq d$ iff $d \neq 0$ and $\| P \| \doteq \{ d \}$, and 0 otherwise;
(D.28) $\| E a \| \doteq 1$ iff $\| a \| \neq 0$
(D.29) $\| a \Pi b \| \doteq 1$ iff $\| a \| \neq 0$ and $\| a \| \leq \| b \|$
(D.30) $\| a T b \| \doteq 1$ iff $\| a \|, \| b \| \neq 0$ and $h(\| a \|) \leq h(\| b \|)$;
(D.31) $\| a \triangleright b \| \doteq 1$ iff $\| b \| \neq 0$ and $\| a \| = h(\| b \|)$;
(D.32) $\| * P \| \doteq \| P \| (\text{the complete } \Lambda \text{-subsemilattice generated by } \| P \|)$;

A number of semantical facts can now be derived that give truth and denotation conditions for all the defined symbols of LPM.

(49) $\| P \subseteq Q \| \doteq 1$ iff $\| P \| \subseteq \| Q \|$ (T.1)
(50) $\| P = Q \| \doteq 1$ iff $\| P \| = \| Q \|$ (T.1)
(51) $\| P \cup Q \| \doteq \| P \| \cup \| Q \| (T.1)
(52) $\| a \| \doteq \{ \| a \| \}$ (T.1)
(53) $\| a = b \| \doteq 1$ iff $\| a \|, \| b \| \neq 0$ and $\| a \| = \| b \| (T.1)
(54) $\| a \sim b \| \doteq 1$ iff $\| a \|, \| b \| \neq 0$ and $h(\| a \|) = h(\| b \|)$ (T.1)
(55) $\| A a \| \doteq 1$ iff $\| a \| \in A (T.1)$
(56) $\| M p a \| \doteq 1$ iff $\| a \| \in D (T.1)$
(57) $\| * P \| \doteq \{ x \in E \mid \exists x \subseteq \| P \| \land x \neq \emptyset \land x = \sup_{a \subseteq x} P \} (T.1)$
(58) $\| * P \| = \| * P \| \setminus A (T.1)$
(59) $\| * P \| \doteq \{ x \in D \mid x \leq \sup_{a \subseteq x} h(\| P \|) \} \doteq \{ x \in D \mid \exists y \in \| * P \| \text{ such that } x \leq h(y) \} (T.1)$
(60) \[ \| \sigma x P x \| \triangleq \sup_P \| P \| \], with \( \sup_P \emptyset \triangleq 0 \)

(61) \[ \| \sigma^* x P x \| \triangleq \| \sigma x P x \| \text{ if } \| P \| \geq 2 \text{ elements, and } 0 \text{ otherwise} \)

(62) \[ \| a \uplus b \| \triangleq \| a \| \uplus \| b \| \]

(63) \[ \| \mu x P x \| \triangleq \sup_m \| b \| \text{ if } \| P \| \neq \emptyset \text{, and } 0 \text{ otherwise} \]

(64) \[ \| a + b \| \triangleq \#(\| a \| \uplus \| b \|) \text{ if } \| a \| \neq 0 \neq \| b \| \]

Let the notion of \textit{truth} in a given model be defined in the usual way. A formula is \textit{valid} if it is true in every model.

I now give, in a loose order, a list of theorems of LPM that come out valid under the above interpretation. I may mention, in particular, the \textit{homogeneous reference properties} (T.11), (T.12), and the \textit{existence and identity criteria} for the \( \sigma^* \) and \( \mu \)-terms, (T.16)–(T.21).

(T.1) \[ a = b \iff a \Pi b \wedge b \Pi a \]

(T.2) \[ a \Pi b \rightarrow a \uplus b \]

(T.3) \[ a = b \rightarrow a \uplus b \wedge b \uplus a \]

(T.4) \[ \text{Equ}(=) \wedge \text{Equ}(\sim) \]

(T.5) \[ P (\Pi) \wedge P (\bar{\Pi}) \]

(T.6) \[ \text{Distr}(P) \rightarrow \land x (P x \rightarrow \neg \sigma x P x) \]

(T.7) \[ P \subset \ast P \]

(T.8) \[ \land x (A t x \rightarrow (P x \rightarrow \ast P x)) \]

(T.9) \[ \land x (\ast P x \rightarrow x \Pi x \uplus y (P y \uplus \Pi x)) \]

(T.10) \[ \text{Distr}(P) \rightarrow \land x \land y (\ast P x \wedge x \Pi y \rightarrow \ast P x) \]

(T.11) \[ \land x \land y (\ast P x \wedge \ast P y \rightarrow \ast P x \uplus y) \]

(T.12) \[ \land x \land y (P x \wedge P y \rightarrow P x + y) \text{ for } P \in \text{MT} \]

(T.13) \[ M (P) \rightarrow P \subset \ast P \]

(T.14) \[ M \sigma a \rightarrow (\ast P a \iff \forall y (\ast P y \wedge a \Pi y)) \]

(T.15) \[ \forall x P x \wedge P \subset Q \rightarrow \ast Q (\sigma x P x) \]

(T.16) \[ \forall x P x \iff \exists \sigma x P x \]

(T.17) \[ \forall x P x \iff \exists \mu x P x \]

(T.18) \[ \exists \sigma^* x P x \iff \forall x \forall y (P x \wedge P y \wedge x \neq y) \]

(T.19) \[ Q P \rightarrow \sigma x P x = \sigma x Q x \]

(T.20) \[ \text{Distr}(P) \rightarrow \text{Distr}(Q) \rightarrow (\sigma x P x = \sigma x Q x \iff P = Q) \]

(T.21) \[ P = Q \rightarrow \mu x P x = \mu x Q x \]

(T.22) \[ \forall \frac{a}{a} = a \iff x (x = a) \]

(T.23) \[ M \sigma a \rightarrow a = \mu x (x = a) \]

(T.24) \[ \mu x P x = \mu x (P x \wedge \forall x \forall y (P y \rightarrow y \Pi x)) \text{ for } P \in \text{MT} \]

(T.25) \[ a \sim b \rightarrow \mu x (x = a) = \mu x (x = b) \]

(T.26) \[ \mu x P x = \mu x (x \Rightarrow \mu x P x) \]

(T.27) \[ M \mu x P x \]

(T.28) \[ a \Rightarrow b \rightarrow M \sigma a \wedge a \sim b \]

(T.29) \[ \mu x P x \sim \sigma x P x \]

(T.30) \[ \sigma x P x = \sigma x Q x \rightarrow \mu x P x = \mu x Q x \]

(T.31) \[ \land x \land y (x \Pi y \leftrightarrow x + y = y) \]

(T.32) \[ \land x \land y (x \Pi y \leftrightarrow x + y = y) \]
3 Applications to Montague Grammar

Let the basic syntax be as in PTQ. The category CN of common noun phrases has to be subcategorized into MCN (mass noun phrases), SCN (singular count noun phrases), and PCN (plural count noun phrases). The quantifiers discussed are $a$, $\exists x$, some, the, all the, every, all; they are of category T/CN with suitable subcategorizations (term formation is done compositionally). We have the obvious plural rule

$$(65) \quad \zeta \in P_{SCN} \Rightarrow \zeta_{pl} \in P_{PCN}$$

where $\zeta_{pl} := \text{hand}$ becomes $\zeta_{pl} := \text{hands}$, $\zeta := \text{child}$, $\zeta_{pl} := \text{children}$, etc.

The logic of PTQ TITL, is extended as follows. TITL', or extended TITL, contains as new symbols all the special symbols of LPM. Additional meaningful expressions are:

(i) $*\zeta, *^\alpha \zeta, *^\tau \zeta \in ME_\rho$, and $*^\tau \zeta \in ME_\rho$ for $\tau = e \in Type$, $\rho = \tau$ or $\tau$, $\zeta \in ME_\rho$;

(ii) $\sigma \mu \phi, \sigma^* \mu \phi, \sigma^* \mu \phi \in ME_\tau$, and $\mu \phi \in ME_\tau$ ($\tau \in Type, u \in Var_\tau$, $\phi \in ME_\tau$);

(iii) $\alpha \Pi \beta \in ME_\tau$ and $\alpha \oplus \beta \in ME_\tau$ ($\tau \in Type, \alpha, \beta \in ME_\tau$); (iv) $\alpha \lor \beta$, $\alpha \land \beta$, $\alpha \rightarrow \beta, \beta \rightarrow \alpha \in ME_\tau$.

The semantics for TITL' has to specify the “boolification” of the sets of possible denotations since the i-sum operation applies to expressions of any type (m-sums are formed on level $e$ only). Let $E$ be a c.a. Boolean algebra, $I, f$ sets, and $K = I \times f$. A Boolean hierarchy of possible denotations over $E$ with respect to $K$ is a family of sets $(D_e, I, f, k)_{k \in K}$ defined as in PTQ, but endowed with a c.a. Boolean structure for every $D_e$ as follows: (i) The sets $D_e = E$ and $D_f = 2$ are already c.a. Boolean algebras; (ii) $f, g \in D_f$; then: $(f \land g)(x) := f(k) \land g(k)(k \in K)$, similarly for the other operations; (iii) $f, g \in D_f$; then $(f \lor g)(x) := f(x) \lor g(x) (x \in D_e)$, similarly for the other operations. The resulting structures are c.a. Boolean.

An interpretation for TITL' is a 7-tuple $\mathfrak{I}(E, A, D, h, I, f, \cdot | |)$ such that (i) $\langle E, A, D, h \rangle$ is a boosk, (ii) $K = I \times f$ is as in PTQ, and (iii) $\cdot | |$ is the interpretation function that naturally emerges from a combination of TITL and LPM. As rules of truth and denotation provided by an appropriate recursive definition let me mention the following: (i) $\cdot | |^* \zeta := | |^* \zeta$ the characteristic function of $\{x \in D_e | |\langle \zeta \rangle | \cdot | | = 1\} \cup \{\zeta \in ME_a\}$;

(ii) $\cdot | |^* \zeta(k) := | |^* \zeta(k)$ the characteristic function of $\{x \in D_e | |\langle \zeta \rangle | \cdot | | = 1\} \cup \{\zeta \in ME_{at}, k \in K\};$

(iii) $\cdot | |^* \zeta := | |^* \zeta \land | |_\tau \mathcal{A}_\tau$ where $\zeta \in ME_{at}, \alpha$; the set of atoms in $D_e$, $\mathcal{A}_\tau$ the characteristic function of $\alpha$, and $\cdot | |$ the Boolean complement; (iv) $| |^* \zeta(k) := | |^* \zeta(k) \land | |_\tau \mathcal{A}_\tau | |(\zeta \in ME_{at}, \alpha \in K)$; (v) $\cdot | |^* \zeta := | |^* \zeta$ with $sup_\varnothing = 0$, and $| |^* \zeta := | |^* \zeta$ if $| |^* \zeta$ has more than one element, and 0 otherwise; (vi) $| | \alpha \Pi \beta := | | \alpha \Pi \beta$ iff $| | \alpha \Pi \beta$ and $| | \alpha \Pi \beta$ if $| | \alpha$ and $| | \beta$ otherwise; (vii) $| | \alpha \oplus \beta := | | \alpha \oplus \beta$ with $\alpha \oplus \beta$ if $| | \alpha \oplus \beta$ has more than one element, and 0 otherwise; (viii) $| | \alpha \lor \beta := | | \alpha \lor \beta$ with $\alpha \lor \beta$ if $| | \alpha \lor \beta$ has more than one element, and 0 otherwise. The remaining basic symbols are characterized as in LPM.

The notion of distributivity introduced above can be extended to class-denoting expressions of any type. Call $\zeta \in ME_{at}$ distributive if it can be true of atoms of type $\tau$ only:

$$(66) \quad \text{DISTR}(\zeta) \leftrightarrow \land_\tau | |\zeta(x) \rightarrow | |_\tau(x) | |(\zeta \in ME_{at})$$
Most of the basic count nouns like child are taken as distributive, similarly IV phrases like die or see. In accordance with what I did above the translation rule T4 involving distributive expressions has to be such that their translations always enter this rule under the star operator. While this seems to me the correct move from a technical point of view I am aware that it carries the empirical prediction that distributivity is a lexical property.

I now give translations for the quantifiers. Let U be the translation relation, and \( \hat{P} = \lambda y[P(y) \land x[\pi y \rightarrow x^T]P(x)] \).

1. a, \( \emptyset \), some
   \[ \hat{\lambda} \hat{y}[\hat{P}(\hat{y}) \land x\hat{y} \rightarrow x^T]P(x)] \]
2. the
   \[ \hat{\lambda} \hat{y}[\hat{P}(\hat{y}) \land x\hat{y} \rightarrow x^T]P(y)] \]
3. all the
   \[ \hat{\lambda} \hat{y}[\hat{P}(\hat{y}) \land x\hat{y} \rightarrow x^T]P(y)] \]
4. every, all
   \[ \hat{\lambda} \hat{P}(\hat{y}) \land x\hat{y} \rightarrow P(x)] \]

\( \hat{\lambda} \) applies only to singular count noun phrases and \( \phi_\emptyset \) to PCN phrases. The quantifiers some and the, with one and the same translation, apply to both singular and plural phrases. As can be seen from 5 through 8 below it is only the incoming CN phrase which differentiates between the appropriate singular and plural readings. While this is obvious with some, a comment on the transformations under 7 is in order. The conjunct \( \land x[\hat{Q}(x) \rightarrow x^T]y \) in the translation of the asserts maximality which is needed for the intended sum formation in the extension of \( ^*Q \). At the same time, however, it generalizes the usual uniqueness condition for definite descriptions. This comes out when the is applied to a singular count noun like child: The i-part \( x \) of \( y \) in 7 cannot be \( 0 \), and \( y \) is an atom because of DISTR (child'); so \( x \) equals \( y \). I think this is a nice instance of strict compositionality in times in which this principle has come under heavy fire in view of all sorts of recalcitrant data (see below). Finally, 9–11 show that universal quantification is as in LPM where, again, the difference in meaning between the children and all the children is only partly characterized (\( \hat{P} \) being defined in terms of \( ^*T \)).

5. a child
   \[ \hat{P}(x[\text{child}'(x) \land P(x)]) \]
6. some child
   \[ \hat{P}(x[\text{water}'(x) \land P(x)]) \]
7. some water
   \[ \hat{P}(x[\text{child}'(x) \land P(x)]) \]
8. the children
   \[ \hat{P}(x[\text{child}'(x) \land x[\text{child}'(x) \rightarrow x^T]y \land P(y)]) \]

I may mention here that the incorporation of numerals like one, two, three, into the fragment is straightforward. The extension of three men, for instance, is the set of all i-sums in \( |\text{men'}||^n\text{man'}| \) which contain exactly three atoms. Sentences like three men went to mow a meadow receive a translation of the form \( \hat{V}z[(\text{three men}')(z) \land Q(z)] \).
The final topic I want to say something about is the vexing problem of relative clauses with more than one head noun.\textsuperscript{17} Call those structures *hydræ*. Let me consider restrictive relative clauses only. First of all, there is a rather friendly type of *hydræ*, like the following.

(67) the German or Austrian John met yesterday
(68) the cabinet-member and mafioso who was deeply involved in the scandal

PTQ style relativization, i.e. CN modification, takes care of these pets yielding the obvious representations for (67) and (68), respectively:

(67') the ((German or Austrian) such that John met him yesterday)
\[\hat{P}\forall y[\land x[\text{German}' \oplus \text{Austrian}'(x) \land \phi(x) \Leftrightarrow x = y] \land P(y)]\]

(68') the ((Cabinet-member and mafioso) such that he was deeply involved in the scandal)
\[\hat{P}\forall y[\land x[\text{Cabinet-member}' \odot \text{mafioso}'(x) \land \psi(x) \Leftrightarrow x = y] \land P(y)]\]

Here, \(\phi\) and \(\psi\) are the translations of the relative clauses of (67) and (68), respectively; furthermore, \(\zeta \oplus \eta \equiv \lambda x[\zeta(x) \lor \eta(x)]\) is the Boolean join and \(\zeta \odot \eta \equiv \lambda x[\zeta(x) \land \eta(x)]\) the Boolean meet of \(\zeta\) and \(\eta\). This case is simple because CN conjunction does not lead to plural structures, and there is only one determiner present.\textsuperscript{18}

Next, let us conjoin two singular count nouns to form a plural phrase. Examples are *man and woman, boy and girl, husband and wife, landlord and tenant*. Such phrases can be true of a sum of two individuals one being of the first and the other being of the second kind. An appropriate translation, therefore, is of the form

(69) \( (\zeta \text{ and } \eta) \ U \ \lambda z \forall x \forall y[\zeta'(x) \land \eta'(y) \land z = x \oplus y] \)

With this sentences like *John and Mary are husband and wife* can be handled in a first approximation while the non-symmetric features of this sentence have to be taken care of by other clues. This is because the predicate (69), by itself, is necessarily symmetric reflecting the fact that pluralization has the force of group formation which typically gives rise to symmetric constructions.\textsuperscript{19} Such is the following CN phrase:

(70) boy and girl who dated each other

With (69), the standard PTQ rule yields for this:

(71) \( \lambda z \forall x \forall y[\text{boy}'(x) \land \text{girl}'(y) \land z = x \oplus y \land \text{dated-each-other}'(z)]\)

Now term formation with one and the same determiner for both head nouns is again standard if we decide to have it distributed over the two nouns in a kind of copying process. So, for instance, *a boy and a girl who meet* becomes

(72) \( a, ((\text{boy and girl}) \text{ such that they meet}) \ U \ \hat{P}\forall z[\forall x \forall y[\text{boy}'(x) \land \text{girl}'(y) \land z = x \oplus y \land \text{meet}'(z)] \land P(z)]\)
In a further step we admit conjunction of plural nouns. This move doesn’t really add any new difficulty so I can write down an example immediately. Let $\theta$ be the translation of the phrase students and professors who had met in secret:

$$(73) \quad \theta \equiv \lambda z \forall x \forall y [\text{student}'(x) \land \text{professor}'(y) \land z = x \oplus y \land \text{had-met-in-secret}'(z)]$$

Adding a determiner as before, say the definite article, we get the term

$$(74) \quad \text{the students and the professors who had met in secret } \quad \text{U} \quad \hat{P} \forall z [z = \sigma* x \theta(x) \land P(z)]$$

The present conception of conjunction of two CN phrases $\zeta$ and $\eta$ is such that the extension of $(\zeta \text{ and } \eta)$ contains i-sums of objects which are of kind $\zeta$ and $\eta$, respectively. Unlike the Boolean join this carries the presupposition that both sets of objects be non-empty. I think this is as it should be. Notice that in this case the overall sum is the same as the one which is formed in terms of the Boolean join, i.e. we have $\sigma(x'(\zeta \text{ and } \eta))(x) = \sigma(x'(\zeta \text{ and } \eta))(x)$. What is not captured in either approach, however, is the proper pair reading which is dominant in sentences like landlords and tenants who hate each other will always find something to argue about. I find these constructions hard to treat at the moment though I am confident that they will finally lend themselves to an analysis which is compatible with the present framework. What is needed here is some notion of ordered pair in the object language, apart from the symmetrical sum operator. It is only with such an additional instrument, it seems to me, that we can, in a close-to-language treatment, attack the most dangerous hydra lurking in the realm of pair reading and branching quantification.

The last type of hydra I want to address myself to shows up in a sentence like the following:

$$(75) \quad \text{all of the students and some of the professors who had met in secret were arrested after the coup d'état.}$$

From a purely syntactic point of view it might look natural to try a T-S-analysis here (Editors’ note: This means an analysis in which restrictive relative clauses are attached at the term phrase level rather than at the common noun phrase level): first conjoin the two unrelativized terms all the students and some of the professors and then modify the conjunct by the relative clause. However, I agree with Jansen (1981) in his sceptical attitude towards the semantical soundness of this approach in general, and the present case, I think, is apt to confirm these doubts. Even a generalized quantifier account along the lines of Barwise and Cooper (1981) is not easy to give because the set of properties denoted by the subject term of (75) is not the obvious intersection of two property sets. The point is that the collective predicate meet-in-secret does not distribute over the conjuncts.

Now there is no reason to leave the approach taken so far, it seems to me. We already have a clear understanding of the meaning of the plural CN phrase students and professors who had met in secret which is expressed by the $\lambda$-term $\theta$ of (73). So in terms of this, what is the meaning of the above NP? Well, the set of properties that all the students of the group of students and professors who had met in secret and some of its professors share. This set can be formally represented in TTTL. From the above
group of individuals we have to pick out again the students and the professors, respectively, which is done by means of the Boolean meet \(\odot\); we can then write (with \(\alpha = \text{'student'}\) and \(\beta = \text{'professor'}\)):

\[
(76) \quad \text{all the students and some of the professors who had met in secret} \quad \bigcup \quad \bar{P}( x \wedge y \cdot x(\alpha \odot \theta)(x) \rightarrow P(y) \big) \wedge \bigvee \big(y(\beta \odot \theta)(y) \wedge P(y) \big)
\]

In the syntax, then, we have to introduce the determiners *all* and *some* simultaneously. I hope that a rule to this effect does not look too outlandish to the eye of the syntactician.

Let me summarize what I consider to be the virtues of the present approach to plurals and mass terms. (1) The logic of plurals and the logic of mass terms share a common lattice structure, the only difference being that the former leads to an atomic structure while the latter does not. (2) By means of the star operator pluralization and term formation involving plural constructions can be treated compositionally. (3) Plural terms (*the cards*) and collective terms (*the deck of cards*) are equivalent in that they are interchangeable in invariant contexts; this does not make them coreferential, however, in contrast to systems of the reductionist lot. (4) Collective predication becomes possible in a unified way accounting for the fact that many predicates (e.g. *carry*) are not marked with respect to distributivity and can, therefore, have mixed extensions.

**Notes**

The basic idea leading to the present approach grew out of a seminar on the semantics of mass terms which I held in the summer of 1980. Meanwhile I had the opportunity to discuss various stages of the paper with a number of people whom I wish to thank here for helpful comments, hints, and criticisms as well as for general support. Among these I should like to mention in particular Ulrich Blau, Paul Gochet, Barbara Partee, Christian Rohrer, Peter Staudacher, Arnim v. Stechow, Alice ter Meulen, Matthias Varga v. Kibéd, and Dietmar Zaefferer.

1 Massey’s example; see Massey (1976).
2 See Quine (1960, p. 91); Bunt (1979).
3 The main source for mass terms is Pelletier (1979); furthermore, see Bennett (1980), Bunt (1979), and ter Meulen (1980, 1981). For the treatment of plurals and collective terms, see Massey (1976), Burge (1977), Blau (1979), Hoepelman and Rohrer (1980), and Scha (1981). The parallelism referred to is explicitly expressed in Bunt (1979).
4 Recently, Keenan and Faltz (1978) and Keenan (1981) advanced a “Boolean approach” to the semantics of natural language. I feel very sympathetic with this enterprise which, unfortunately, I became aware of only a year ago (January 1981). It is reassuring to see similar techniques be successfully applied in other areas of semantics, too. I have to defer a concrete evaluation of these ideas to another occasion.
5 On this point I agree with ter Meulen (1981), I think. But I do not follow her in the conclusions she draws from this observation. The inherent lattice structure is independent of the philosophical motives that gave rise to the construction of mereological systems. For the role of nominal mass noun denotations see the remarks below, also note 11.
6 I guess that in German, with *die Karten vs. das Kartenspiel*, the point might come out more clearly.
7 The point was apparently made first by David Kaplan as Bennett (1980) reports.
8 This is the situation originally analyzed by Blau in his 1979 paper.
9 This is Burge’s example, see Burge (1977).
This approach is the traditional one. In one form or another it can be found, for instance, in Bennett (1974), Hauser (1974), von Stechow (1980), Hoepelman and Rohrer (1980), Scha (1981). Contributions to the problem of substance names can be found in Pelletier (1979) (in particular, Parsons (1979)), Bunt (1979), and ter Meulen (1980, 1981). Let me comment on the latter work, which is formulated in a Montague framework. The few remarks I made here will make it evident that I fully agree with ter Meulen in that nominal mass nouns cannot be reduced to predicative mass nouns. But for this very reason I fail to see any cogent argument for the kind of denotation ter Meulen wants to assign to these terms at a given reference point (i.e. intension functions denoting in each world the set of concrete quantities of the substance in question). As it turns out, the arguments she puts forth in ter Meulen (1981) really lend support only to the first, the critical, point (viz. that reduction is impossible). But what she then goes on to call a nominal mass noun’s “extensional reference to an intensional object” (viz. the intension function referred to above) seems to me both syntactically and semantically misguided. For the inevitable doubling of syntactic rules is certainly unwelcome, to begin with. But what is more, those intension functions, even when lifted to still another intensional level as ter Meulen wants to have it, are simply not well motivated as substance name denotations. The statement, for instance, that two fictional substances can be differentiated (op. cit., p. 438) is not compatible with the principle of rigid designation introduced earlier (op. cit., p. 424). More generally, there are no rules that could justify intuitively valid inferences from contexts involving nominal mass nouns to contexts with their corresponding predicative terms – it is my view, anyway, that such inferences are not based on pure logic alone. I conclude from this that the problem of nominal mass nouns is best approached in a spirit of logical abstinence. Nominal mass nouns denote abstract entities, to be sure, and as such they are names of individuals just like John, Munich, and the rest. Beyond this minimal account things become notoriously vague.

References


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Let me begin by noting that assertion is not just something that people do, it represents the possibility of a situation that could have happened with their own participation included sometimes the realisation of a situation is made, for example, by the acts of asserting. Thus, the attitudes of the hearer in understanding depend on it.

My aim in this paper is to explore these truisms in more detail. My main theory of speech act is based on the mentioned truisms, but I will make the assumption that they are not sufficient to draw the conclusion, I hope, that the modest claim.

Three notions are involved: proposition presupposition and relevance. The notion of relevance is most important to my theory. However, I will not go into the details of the logic of expressing. I will say what conversations are called relevant conversations and what conversations are not. Various differences in the ways in which those activities are meaningful for them, is that...