A “De-Fregean” Semantics for Modified and Unmodified Numerals*

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DRAFT: July 20, 2013

Abstract Nouwen (2010) shows that an adequate semantics for modified numerals must explain why certain modifiers but not others trigger ignorance/uncertainty inferences regarding the quantity characterized by the numerical expression, and must also explain a complex set of patterns of interactions with modals. My goal in this paper is first to show that Nouwen's account of the patterns, while insightful, runs into a number of empirical problems; and second to show that these problems are avoided by a “de-Fregean” semantic analysis of both modified and unmodified numerals as generalized quantifiers over degrees. I begin with an overview of the facts discussed by Nouwen, a presentation of his proposals, and an illustration of the challenges that it faces. I then introduce the de-Fregean analysis of unmodified number words, motivate it by showing how it explains the scalar inferences of sentences containing them (lower-bounded vs. upper-bounded vs. “two-sided” readings), and then show how extending it to modified numerals supports a pragmatic account of ignorance/uncertainty inferences and correctly derives the observed scopal interactions of modified numerals and modals. I conclude with a comparison to focus-based analyses.

1 Two classes of numeral modifiers

Nouwen (2010) introduces a challenge for the semantic and pragmatic analysis of modified numerals involving the presence vs. absence of ignorance/uncertainty inferences (see also Geurts and Nouwen 2007). Certain modifiers, such as *at least/at most, minimally/maximally, n or more* or *up to*, which Nouwen calls “Class B” modifiers, trigger an inference that the actual quantity of things which satisfy the property expressed by the (relevant parts of) the rest of the sentence is unknown or uncertain. Thus (1a-b) sound strange because the ignorance/uncertainty inference clashes with our common knowledge that hexagons have six sides.

(1) a. # A hexagon has at least/minimally four sides.
    b. # A hexagon has at most/maximally ten sides.

*Acknowledgments.

These modifiers contrast with *more than/fewer than* and *over/under*, which Nouwen labels "Class A" modifiers. (2a-b) appear to provide exactly the same information as the corresponding sentences in (1), yet do not give rise to ignorance/uncertainty inferences, and so are perfectly acceptable as characterizations of the sidedness of a hexagon (cf. Geurts and Nouwen 2007).

(2) a. A hexagon has more than/over three sides.
b. A hexagon has fewer than/under eleven sides.

Class B modifiers also contrast with the absence of modification: all of the examples in (3) are acceptable.

(3) a. A hexagon has six sides.
b. A hexagon has four sides.
c. A hexagon has ten sides.

Note in particular that (3c) is acceptable even though it is false, and (3b) is acceptable regardless of whether it is understood in a "two-sided," exact-quantity way (in which case it is false) or in a "one-sided," lower-bounded way (in which case it is true).

Nouwen's account of this pattern involves three parts. The first is the hypothesis that numerals (both modified and unmodified) saturate a degree position in the nominal projection which, following Hackl (2000), he takes to be provided by a parameterized cardinality determiner *many*. This determiner comes in two versions: the "weak" version in (4a), which involves regular existential quantification over plural individuals; and the "strong" version in (4b) which adds a uniqueness requirement (indicated by "!").

(4) a. $[\![many_w]\!] = \lambda n \lambda P \lambda Q.\exists x[Q(x) \wedge P(x) \wedge \#(x) = n]$
b. $[\![many_s]\!] = \lambda n \lambda P \lambda Q.\exists! x[Q(x) \wedge P(x) \wedge \#(x) = n]$

The second part involves a semantic distinction between modified and unmodified numerals. The latter are singular terms, and denote values in the range of the measure function '#' encoded by *many*, which Nouwen (with Hackl) assumes to be elements in the domain of degrees (see also Cresswell 1976; Krifka 1989).[1] Depending on which version of *many* is chosen, the result is either lower-bounded

[1]Note that # is not, strictly speaking, a cardinality function, but rather gives a measure of the size of the (plural) individual argument of the noun in "natural units" based on the sense of the noun (Krifka 1989; Salmon 1997). If this object is formed entirely of atoms, then # returns a value that is equivalent to a cardinality. But if this object contains parts of atoms, then # returns an appropriate fractional or decimal measure. For the purpose of this paper, we can assume that the range of # has at least the structure of the real numbers (Fox and Hackl 2007).

(*many*$_w$) or two-sided truth conditions (*many*$_s$), as illustrated in (5), the two parses of (3b). (For simplicity I take the indefinite subject here to denote an individual, hexagon-kind, here symbolized by **hex**.)

(5) a. $\exists x[\textbf{have}(x)(\textbf{hex}) \wedge \textbf{sides}(x) \wedge \#(x) = 4]$ TRUE
b. $\exists! x[\textbf{have}(x)(\textbf{hex}) \wedge \textbf{sides}(x) \wedge \#(x) = 4]$ FALSE

Modified numerals, on the other hand, denote generalized quantifiers over degrees. Nouwen assumes a fairly standard semantics for Class A modifiers, which builds on work in comparatives: *more than* and *fewer than* have the denotations in (6a-b).

(6) a. $[\![more\ than]\!] = \lambda m \lambda P_{\langle d,t\rangle}.max\{n \mid P(n)\} > m$
b. $[\![fewer\ than]\!] = \lambda m \lambda P_{\langle d,t\rangle}.max\{n \mid P(n)\} < m$

These denotations give the intuitively correct truth conditions for (2a-b) in (7a-b): the maximum number of sides that a hexagon has is greater than three and fewer than eleven, respectively. Here there is no truth conditional difference here between *many*$_w$ and *many*$_s$, so I just give meanings for the former.

(7) a. $max\{n \mid \exists x[\textbf{have}(x)(\textbf{hex}) \wedge \textbf{sides}(x) \wedge \#(x) = n]\} > 3$
b. $max\{n \mid \exists x[\textbf{have}(x)(\textbf{hex}) \wedge \textbf{sides}(x) \wedge \#(x) = n]\} < 11$

For the Class B modifiers, Nouwen proposes the meaning in (8a) for lower-bound modifiers like *at least*, and the one in (8b) for upper-bound modifiers like *at most*.

(8) a. $[\![at\ least]\!] = \lambda m \lambda P_{\langle d,t\rangle}.min\{n \mid P(n)\} = m$
b. $[\![at\ most]\!] = \lambda m \lambda P_{\langle d,t\rangle}.max\{n \mid P(n)\} = m$

These denotations give the truth conditions in (9a-b) for (1a-b), respectively. For the lower-bound Class B modifiers, it is now crucial that we parse the sentence using *many*$_s$, because *many*$_w$ returns contradictory truth conditions for any numeral greater than 1.

(9) a. $min\{n \mid \exists x![\textbf{have}(x)(\textbf{hex}) \wedge \textbf{sides}(x) \wedge \#(x) = n]\} = 4$
b. $max\{n \mid \exists x[\textbf{have}(x)(\textbf{hex}) \wedge \textbf{sides}(x) \wedge \#(x) = n]\} = 10$

Both (9a) and (9b) introduce two-sided truth conditions: the former requires the minimum unique plurality of sides of the hexagon to be exactly four; the latter requires the maximum unique plurality of sides of the hexagon to be exactly 10. Observing that these truth conditions are equivalent to what we get from corresponding, simpler, bare numeral constructions on their *many*$_s$ parses, Nouwen proposes that principles of blocking rule them out as possible meanings for (1a-b), making the corresponding sentences infelicitous.

There is a way of modifying (9a-b) to derive interpretations for (1a-b) that are distinct from the bare numeral meanings, however, which corresponds to the third part of Nouwen's account of the uncertainty/ignorance inferences associated with Class B modifiers. Nouwen proposes that a silent epistemic possibility modal can be inserted into sentences like those in (1). If the modified numeral takes scope over the modal, we get the truth conditions in (10a-b), which differ crucially from those in (9a-b) in that the sets that are the inputs to the *min* and *max* operators are no longer singletons.

(10) a. $min\{n \mid \diamond\exists x![\textbf{have}(x)(\textbf{hex}) \wedge \textbf{sides}(x) \wedge \#(x) = n]\} = 4$
b. $max\{n \mid \diamond\exists x[\textbf{have}(x)(\textbf{hex}) \wedge \textbf{sides}(x) \wedge \#(x) = n]\} = 10$

(10a) says that the minimum number in the set of unique numbers n such that there is an epistemically accessible world in which a hexagon has n sides is four; (10a) says that the maximum such number is ten. When there is uncertainty about quantity, these sets will contain more than one number (uncertainty entails the existence of epistemically accessible worlds in which hexagons have different numbers of sides). The contribution of the modifiers in such a situation is to express lower- and upper-bounds, respectively, on the relevant sets. In the case of hexagons, there is no actual uncertainty, so the truth conditions are again equivalent to the unmodified forms, the readings are blocked as before, and the the sentences are infelicitous. But in more typical examples, the result will be exactly the uncertainty inferences that we want to derive.[2]

2 Two problems with modals

Although Nouwen's analysis succeeds in deriving the uncertainty inferences associated with Class B modifiers, it faces a number of empirical challenges. Some of these have already been discussed in the literature (see Schwarz, Buccola, and Hamilton 2012); here I want to focus on two problems involving the interaction of Class B modified numerals and modals that raise further challenges which, I believe, point to the need for a new account of the Class A/B distinction.

[2] Alternatively, we can say that a parse with an epistemic modal implicates uncertainty, precisely because without it there is no truth conditional difference to the unmodified form, which then clashes with the fact that everyone knows how many sides a hexagon has; or that uncertainty is a precondition for the "last resort" insertion of the modal in the first place (i.e., a pragmatic presupposition); see Nouwen (2010) for additional discussion. The analysis of modified numerals that I present in Section 3.2 will obviate the need to posit an epistemic modal in these examples.

2.1 Class B modifiers and deontic modals

The first problem with modals is noted by Nouwen himself: the analysis appears to make incorrect predictions about the truth conditions of sentences in which Class B modified numerals are embedded under deontic modals. Nouwen focuses on the case of (11), which is predicted to have the two interpretations shown in (11a-b), depending on whether the numeral takes scope below or above the modal. (In the examples to follow, I use "⊗" to indicate an unavailable reading that Nouwen's analysis can derive via blocking, and "*" to indicate an unavailable reading that Nouwen's analysis cannot derive by blocking.)

(11) You are required to register for at least three classes.

a. $\otimes\, \Box[min\{n \mid \exists! x[\textbf{reg}(x)(\textbf{you}) \wedge \textbf{classes}(x) \wedge \#(x) = n]\} = 3]$

b. $*\; min\{n \mid \Box\exists! x[\textbf{reg}(x)(\textbf{you}) \wedge \textbf{classes}(x) \wedge \#(x) = n]\} = 3$

On the narrow scope numeral interpretation, (11) is predicted to be true just in case every deontically accessible world is one in which the minimum unique number of classes registered for is three, which entails registration in no more and no fewer than three classes. (11) clearly does not have such an interpretation, though the corresponding bare numeral sentence does (*You are required to register for three courses*), so the absence of this reading could be explained in terms of blocking. The problem is that the truth conditions associated with the wide scope numeral interpretation are identical: (11b) says that three is the minimum n such that in every deontically accessible world there is registration in a unique number of classes equal to n, which again entails registration in no more and no fewer than three classes. So this reading should be blocked as well, and the sentence should be unacceptable.

Of course, (11) is perfectly acceptable, and its truth conditions are clear: registration in one or two (or zero) classes is not allowed; registration in three is a must; registration in more than three is an option. As Nouwen observes, this meaning is represented by the formula in (12), in which the numeral scopes over an existential modal operator.

(12) $min\{n \mid \Diamond\exists! x[\textbf{reg}(x)(\textbf{you} \wedge \wedge \textbf{classes}(x)\#(x) = n]\} = 3$

Nouwen does not provide an explanation for how or why this shift from universal to existential modal force occurs, or why there is not another reading in which the modified numeral scopes below the modal (which would give the wrong truth conditions), though he notes that it appears to be a systematic feature of universal modal statements expressing minimum requirements (cf. von Fintel and Iatridou 2007), showing up also in relative clause structures like (13).

(13) The minimum number of classes that you need to register for is three.

The analysis that I will present in Section 3.2 will derive the correct meanings for these sentences without invoking blocking effects or necessitating a change in modal force.

Nouwen focuses mainly on the interaction of minimizing Class B modifiers with universal modals, but when we look at the full range of possibilities — pitting modal force against maximality/minimality — we see that the problems go beyond the examples just considered. Consider first the interaction of maximizing modifiers and universal modals. The following example is predicted to have the two readings in (14a-b) depending on whether the numeral scopes below or above the modal, neither of which are correct.

(14) You are required to register for at most three classes.

a. $\otimes\, \Box[max\{n \mid \exists x[\textbf{reg}(x)(\textbf{you}) \wedge \textbf{classes}(x) \wedge \#(x) = n]\} = 3]$

b. $*\ max\{n \mid \Box[\exists x[\textbf{reg}(x)(\textbf{you}) \wedge \textbf{classes}(x) \wedge \#(x) = n]]\} = 3$

(14a) says that every deontically accessible world is such that there is a maximum of three courses registered for. If three is the maximum in every world, then there are no worlds with registration in one or two classes, and no worlds with registration in four or more classes; i.e., there is registration in exactly three courses in all worlds. This is not a reading of (14), though Nouwen could again appeal to blocking here.

(14b) is a bigger problem. This logical representation says that three is the maximum n such that in every world, there is registration for at least n classes. This disallows registration in one or two classes, and allows registration in three or more classes; in other words, this is precisely the meaning that we failed to derive without modal shifting for (14)! It is, moreover, one of the meanings associated with the corresponding bare numeral sentence, namely the lower-bounded one (Geurts 2006). The prediction, then, appears to be that (14) should be infelicitous, which is clearly the wrong result. In fact, the sentence is ambiguous, but neither reading is captured by (14a-b). One reading forbids enrollment in four or more classes. The other reading is weaker: registration in one to three classes is required, but registration in four or more is not ruled out. Neither of these readings can be derived from Nouwen's semantics for *at most*.

Next consider minimizing Class B modifiers and existential modality. The following sentence should have the readings represented in (15a-b).

(15) You are allowed to register for at least three classes.

a. $\otimes\, \Diamond[min\{n \mid \exists! x[\textbf{reg}(x)(\textbf{you}) \wedge \textbf{classes}(x) \wedge \#(x) = n]\} = 3]$

b. $*\ min\{n \mid \Diamond[\exists! x[\textbf{reg}(x)(\textbf{you}) \wedge \textbf{classes}(x) \wedge \#(x) = n]]\} = 3$

(15a) says that there is a deontically accessible world with registration in exactly three classes. This is certainly compatible with (15), but it is also one of the meanings assigned to the corresponding bare numeral sentence, and so presumably should be blocked. (15b) says that three is the minimum n such that there is a deontically accessible world with exactly n registered-for classes. This allows for worlds with registration in more than three classes, but it rules out worlds with registration in one or two classes. The expectation, then, is that (15) should have the meaning that we wanted to derive for (11). This is actually not surprising, given Nouwen's suggestion that the universal modal force is converted to existential modal force in that example, but it is a problem for the analysis of (15), since it cannot be understood in this way.

Finally, consider the interaction of existential modals and maximizing Class B modifiers.

(16) You are allowed to register for at most three classes.

a. $\otimes\, \Diamond[max\{n \mid \exists x[\mathbf{reg}(x)(\mathbf{you}) \wedge \mathbf{classes}(x) \wedge \#(x) = n]\} = 3]$

b. $max\{n \mid \Diamond[\exists x[\mathbf{reg}(x)(\mathbf{you}) \wedge \mathbf{classes}(x) \wedge \#(x) = n]]\} = 3$

If the numeral scopes below the modal, as in (16a), the predicted meaning is that registration in exactly three courses is allowed. This is a weak meaning, because it does not rule out registration in four or more courses (or in one or two courses). It does not appear to be a possible reading of (15), which explicitly puts an upper bound on the number of registered-for courses, though it is a meaning of the corresponding bare numeral sentence, so Nouwen can appeal to blocking here.

If the numeral scopes above the modal, as in (15a), the truth conditions state that three is the maximal n such that there is a deontically accessible world with registration in n classes. This is precisely what (15) means, so it appears that Nouwen's analysis derives the correct result for this case. However, it is also the case that the bare numeral sentence (17) can be used to convey precisely the same information, that is that registration in more than three courses is not allowed:

(17) You are allowed to register for three courses.

Why, then, is there no blocking effect here? The answer is that, given Nouwen's assumptions about bare numerals, this meaning must be derived by implicature. (18a-b) show the truth conditions that Nouwen predicts for (17), which differ in whether the propositional argument of the modal has a one-sided (*many*$_w$) or two-sided (*many*$_s$) understanding of the numeral construction.

(18) a. $\Diamond[\exists x[\mathbf{reg}(x)(\mathbf{you}) \wedge \mathbf{classes}(x) \wedge \#(x) = n]]$

b. $\Diamond[\exists! x[\mathbf{reg}(x)(\mathbf{you}) \wedge \mathbf{classes}(x) \wedge \#(x) = n]]$

(18b) is equivalent to (16a), simply entailing the existence of a world with registration in exactly three classes, but not ruling out worlds with registration in other numbers of classes. (18a) is even weaker, since we only know that registration in some number of classes of at least size three is acceptable. But precisely because both readings are so weak, they can be pragmatically strengthened by a Quantity implicature of the sort in (19).

(19) $\forall n > 3\neg[\Diamond[\exists(!)x[\textbf{reg}(x)(\textbf{you}) \wedge \textbf{classes}(x) \wedge \#(x) = n]]]$

Adding (19) to the semantic content of (17) (on either interpretation) derives a meaning that is equivalent to (16b), but since (19) is an implicature, this evidently does not block (16b).

That said, if we were to find evidence that the content in (19) were part of the truth conditions of (17), we *would* expect blocking, and Nouwen's account of the meaning and acceptability of (16) would no longer be tenable. In section 3.1, I will present such evidence, and provide a semantic analysis of unmodified numerals that derives upper-bounded readings compositionally. If this analysis is correct, then Nouwen's account of (16) does not go through.

2.2 Class B modifiers and epistemic modals

I now turn to interactions with epistemic modals, which are not discussed by Nouwen. First consider the following two examples, which contain minimizing and maximizing Class B modifiers, respectively.

(20) a. Chicago has at least 200 distinct neighborhoods.
b. Chicago has at most 3,000,000 residents.

As expected, these sentences are acceptable only if there is some uncertainty about the exact number of neighborhoods in Chicago, and about the exact population in Chicago; when uncertainty is eliminated, the sentences are infelicitous:

(21) # Thanks to the detailed information provided in this census report, I know precisely how many distinct neighborhoods Chicago has, and its exact population: it has at least 200 distinct neighborhoods, and it has at most 3,000,000 residents.

Recall from the discussion in Section 1 that Nouwen derives the uncertainty inferences (20) by scoping the modified numeral over an unpronounced epistemic possibility modal, which is itself inserted into the structure to bypass the blocking effect that would otherwise arise from the truth-conditional equivalence between the non-modalized versions of (20a-b) and the corresponding sentences with bare numerals. This account of the uncertainty inference would seem to predict that variants of (20) with *overt* epistemic modals, such as (22a-b), should have parallel meanings.

(22) a. Chicago might have at least 200 distinct neighborhoods.
b. Chicago might have at most 3,000,000 residents.

But this is not the case: (22a-b) are not synonymous with (20a-b), and in particular, they lack interpretations in which the modified numeral takes scope over the modal. (22a-b) can only be understood to say that it is epistemically possible that the minimum number of Chicago neighborhoods is 200 and that it is epistemically possible that its maximum population is 3,000,000; these sentences cannot be understood to make the stronger claims associated with (20a-b) which rule out epistemically accessible worlds in which Chicago has fewer than 200 distinct neighborhoods or more than 3,000,000 residents.

In fact, the absence of a wide-scope interpretation for the modified numeral in these examples is expected: previous work on degree quantifiers — expressions of type $\langle\langle d,t\rangle,t\rangle$, which includes the class of modified numerals in Nouwen's analysis — has shown that they may take scope higher than root modals but not higher than epistemic modals (Heim 2000; Schwarzschild and Wilkinson 2002; Büring 2007; Krasikova 2010; Alrenga and Kennedy in press). This is illustrated by the following pair of examples, which involve the negative differential degree quantifier *no*:

(23) a. Kim can jump no higher than Lee.
b. Kim might jump no higher than Lee.

(23a) has a reading in which it asserts that Kim lacks the ability to jump higher than Lee (and implicates that Kim can jump at least as high as Lee). Alrenga and Kennedy (in press) show that this reading arises when *no* takes scope over the ability modal, deriving the truth conditions stated informally in (24).

(24) $\{d \mid \Diamond[\textbf{Kim jumps } d\textbf{-higher than Lee in } w]\} = \emptyset$

When the relevant accessibility relation is based on Kim's abilities, this derives the observed interpretation of (23a). What is important for us here is that (23b) does not have a corresponding wide-scope meaning for *no*. Such a meaning would say that the set of degrees d such that there is an epistemically accessible world in which Kim jumps d-higher than Lee is empty, which is another way of saying that there is no evidence that Kim jumped higher than Lee. But (23b) merely says that there is evidence that Kim's jump will not be higher than Lee's, which is a much weaker claim, and is precisely the one that we get when *no* stays within the scope of the modal:

(25) $\Diamond[\{d \mid \textbf{Kim jumps } d\textbf{-higher than Lee in } w\} = \emptyset]$

If it is generally the case that degree quantifiers cannot take scope over epistemic modals, then the missing readings of (22) are expected. On the other hand, if it is generally the case that degree quantifiers cannot take scope over epistemic modals, then Nouwen's account of uncertainty inferences with Class B modifiers faces a serious challenge, since it rests on the idea that precisely such a scopal configuration is created in order to avoid a blocking effect and generate the observed interpretations of sentences like the ones in (20). Indeed, to derive the actual meanings of (22a-b), Nouwen must assume that in addition to the overt modal, a second, silent modal is inserted below the scope of the modified numeral, as shown in (26a-b).

(26) a. $\Diamond[min\{n \mid \Diamond[\exists! x[\textbf{have}(x)(\textbf{chi}) \wedge \textbf{neighborhoods}(x) \wedge \#(x) = n]]\} = 200]$

b. $\Diamond[max\{n \mid \Diamond[\exists! x[\textbf{have}(x)(\textbf{chi}) \wedge \textbf{residents}(x) \wedge \#(x) = n]]\} = 3,000,000]$

In the absence of the silent modal, the truth conditions for (22a-b) would be identical to corresponding sentences with bare numerals, for the reasons described above, and so should be blocked.

There are various ways that Nouwen could respond to this challenge. Perhaps "last resort" insertion of a silent epistemic modal does not interact with whatever constraint blocks wide scope of a degree quantifier relative to an existential modal. Or perhaps the observed readings are not compositionally derived in the first place, but rather involve some sort of "pragmatic" mapping from the compositionally-derived content (which is ruled out by blocking) to the observed meaning. On the other hand, if an alternative analysis avoids these complications entirely, that would certainly be a point in its favor. I now turn to such an analysis.

3 A "de-Fregean" semantics for modified and unmodified numerals

In this section, I present an alternative account of the properties of Class A/B modified numerals, which is designed both to account for both the uncertainty effects that provide the inspiration for Nouwen's proposals and for the interactions with modals that we examined in Section 2. The central thesis is that all numerals — modified and unmodified alike — have (or at least can have) essentially the same meanings: they denote second order properties of degrees, differing only in the kinds of orderings they introduce. In section 3.1, I present my account of unmodified numerals, and then extend the analysis to modified numerals in section 3.2. I argue that the uncertainty inferences of Class B modifiers arise as implicatures because they are less informative than either Class A modified numerals or unmodified numerals, and I show that the analysis avoids the problems with modals that Nouwen's account runs into, deriving the correct meanings for the various scopal possibilities. I finish with a discussion of a set of examples that Nouwen (2010) presents as a challenge for the analysis of Class B modifiers that I will defend, showing how they can be accommodated within the larger theory of numeral meaning that I am advocating.

3.1 Unmodified numerals

For many years, the standard view about the relation between sentence meaning and speaker meaning in assertions involving numerals was the one expressed in the following quote from Horn 1972:

> Numbers, then, or rather sentences containing them, assert lower boundedness — *at least n* — and given tokens of utterances containing cardinal numbers may, depending on the context, implicate upper boundedness — *at most n* — so that the number may be interpreted as denoting an exact quantity. (Horn 1972, p. 33)

Although this view is still encountered in the literature and in introductory semantics and pragmatics courses, a range of studies have appeared since Horn 1972 in both the theoretical and experimental literature (some by Horn himself) which provide compelling evidence that the two-sided, "exactly" understanding of sentences containing numerals is a matter of semantic content, rather than implicature (see e.g. Sadock 1984; Koenig 1991; Horn 1992; Scharten 1997; Carston 1998; Krifka 1998; Noveck 2001; Papafragou and Musolino 2003; Bultinck 2005; Geurts 2006; Breheny 2008; Kennedy 2013b).

The most straightforward way to implement a two-sided semantics for sentences with numerals is to bite the bullet and analyze numerals as quantificational determiners with two-sided meanings, as in (27a) (Koenig 1991; Breheny 2008), or as determiners that are ambiguous between the two-sided meaning in (27a) and the lower-bounded meaning in (27b) (Geurts 2006).

(27) a. $[\![\text{three}]\!] = \lambda P_{\langle e,t\rangle}\lambda Q_{\langle e,t\rangle}. \mid P \cap Q \mid = 3$
b. $[\![\text{three}]\!] = \lambda P_{\langle e,t\rangle}\lambda Q_{\langle e,t\rangle}. \mid P \cap Q \mid \geq 3$

A second way is to adopt the assumptions that I outlined in the previous section: treat bare numerals as singular terms denoting numbers (objects of type d), and assume two versions of Hackl's (2000) parameterized cardinality determiner, $\textit{many}_s$ and $\textit{many}_w$, which differ in whether they introduce simple existential quantification over individuals or existential quantification plus uniqueness. Composition of a numeral with the former derives lower-bounded semantic content; composition with the latter derives two-sided semantic content.

In Kennedy (2013b), however, I develop an alternative analysis of two-sided content which is based on the hypothesis that unmodified numerals, just like modified numerals in Nouwen's (2010) analysis, can denote second order properties of degrees. Specifically, I propose that unmodified numerals can have type $\langle\langle d,t\rangle,t\rangle$, generalized quantifier denotations of the sort shown in (28), in addition to their type d, singular term denotations. (I return to a discussion of the relation between the two meanings below.)

(28) $[\![\text{three}]\!] = \lambda D_{\langle d,t\rangle}.max\{n \mid D(n)\} = 3$

The numeral *three*, on this analysis, is true of a property of degrees if the maximum number that satisfies the property is three. This analysis is inspired by the treatment of numerals as second order properties of individuals considered in Frege 1980 [1884], in which e.g. *three* is true of a property of individuals just in case the number of individuals that the property is true of is three (cf. Scharten 1997), and so I refer to it as a "de-Fregean" semantics for numerals. The central compositional difference between the two accounts is that I treat bare numerals as members of the class of degree expressions and numbers as degrees, and so am able to make full use of all the principles and assumptions of degree syntax and semantics that have been established in work on comparatives, modified numerals and other degree constructions.

In particular, as in Nouwen's analysis, I assume that numerals saturate a degree position in the nominal projection, which could be introduced by Hackl's *many*$_w$ (we no longer need *many*$_s$, as we will shortly see), by a silent/deleted adjectival version of *many* (cf. Landman 2003, 2004), or by the noun itself (Cresswell 1976; Krifka 1989). For present purposes, these distinctions do not matter, so I will adopt the Cresswell/Krifka approach for simplicity, and assume that the nominal and verb combine via Chung and Ladusaw's (2004) "Restrict" rule, with the individual argument of the noun bound by existential closure. The result is that a sentence like (29a) has the truth conditions in (29b) when *three* is interpreted as a de-Fregean quantifier.

(29) a. Kim took three classes.

b. $max\{n|\exists x[\textbf{took}(x)(\textbf{kim}) \wedge \textbf{classes}(x) \wedge \#(x) = n]\} = 3$

According to (29b), (29a) is true just in case the maximal n such that Kim took at least n classes is equal to three, which is false if she took two and false if she took four. These are the two-sided truth conditions.

One of the main advantages of the de-Fregean analysis, in addition to deriving two-sided meanings semantically, is that it explains a pattern of scalar readings first (to my knowledge) observed by Scharten (1997), and illustrated very clearly in experimental work by Musolino (2004). Specifically, although two-sided readings are the default for simple sentences like (29a), one-sided (lower- and upper-bounded) readings appear in a systematic and predictable way when numerals are embedded under (root) modals: lower-bounded readings appear when a numeral is embedded under a universal modal, and upper-bounded readings emerge when a numeral is embedded under an existential modal. The examples in (30) and (31) illustrate the pattern.

(30) a. In Britain, you have to be 17 to drive a motorbike and 18 to drive a car.
b. Mary needs three As to get into Oxford.
c. Goofy said that the Troll needs to put two hoops on the pole in order to win the coin.
d. You must provide three letters of recommendation.
e. You are required to take three classes per quarter.

(31) a. She can have 2000 calories a day without putting on weight.
b. You may have half the cake.
c. Pink panther said the horse could knock down two obstacles and still win the blue ribbon.
d. You are permitted to take three cards.
e. You are allowed to enroll in three classes per quarter.

The de-Fregean analysis derives this pattern as a scopal interaction between numerals and modals. (For independent arguments that bare numerals must be able to take scope separately from the nominals with which they compose in the surface form, see Kennedy and Stanley 2009.)

Consider first the case of universal modals. The sentence in (32) can be interpreted either with the number word inside the scope of the modal, deriving the proposition in (32a), or with the modal inside the scope of the number word, deriving the proposition in (32b).

(32) Kim is required to take three classes.
a. $\Box[max\{n \mid \exists x[\textbf{take}(x)(\textbf{kim}) \wedge \textbf{classes}(x) \wedge \#(x) = n]\} = 3]$
b. $max\{n \mid \Box[\exists x[\textbf{take}(x)(\textbf{kim}) \wedge \textbf{classes}(x) \wedge \#(x) = n]]\} = 3$

(32a) is true just in case every deontically accessible world is such that the maximum number of classes taken by Kim in that world is three. This is the two-sided reading. (32b) is true just in case the maximum number n, such that in every deontically accessible world there is a plurality of classes of at least size n taken by Kim, is three. This entails that the minimum number of deontically acceptable classes is three, which is the lower bounded meaning.

In the case of a sentence with an existential modal like (33), we get exactly the same scopal relations, but the resulting truth conditions are quite different:

(33) Kim is allowed to take three classes.

a. $\Diamond[max\{n \mid \exists x[\textbf{take}(x)(\textbf{kim}) \wedge \textbf{classes}(x) \wedge \#(x) = n]\} = 3]$

b. $max\{n \mid \Diamond[\exists x[\textbf{take}(x)(\textbf{kim}) \wedge \textbf{classes}(x) \wedge \#(x) = n]]\} = 3$

(33a) is the "weak" reading of (33), which merely says that there is a deontically accessible world in which the maximum number of classes taken by Kim is three. (33b), on the other hand, says that the maximum n such that there is a deontically acceptable world in which Kim takes at least n classes is three. On this reading, the sentence is false if there is a deontically accessible world in which Kim takes more than three classes. This is the "strong" reading of (33), and the fact that it is derived compositionally, rather than via a scalar implicature, represents one of the central empirical differences between the de-Fregean analysis and all other approaches to number word meaning, in which such readings can only be derived via implicature.

We have already seen that an analysis of two-sided readings in terms of *many*$_s$ cannot derive upper-bounding readings in examples like (33) semantically, because embedding a *many*$_s$ proposition under an existential modal, as in (34a), only derives the weak reading of (33). Similarly, embedding a Breheny-style two-sided quantificational determiner under the modal, as in (34b), derives only the weak reading.

(34) a. $\Diamond[\exists! x[\textbf{take}(x)(\textbf{kim}) \wedge \textbf{classes}(x) \wedge \#(x) = n]]$

b. $\Diamond[\mid \lambda x.\textbf{classes}(x) \cap \lambda x.\textbf{take}(x)(\textbf{kim}) \mid \ = 3]$

Of course, both of these meanings can be straightforwardly strengthened to produce the strong reading by adding a scalar implicature to the effect of *there is no world w such that the number of classes in w = n, for all n greater than 3*, so the crucial question is whether the upper bounding reading can be retained in environments in which scalar implicatures disappear. If the answer is yes, then we know that it must be derived as a matter of semantic content, and we have evidence in support of the de-Fregean analysis over the traditional alternatives.

Such evidence is discussed in Kennedy (2013b). To set up the example, imagine a situation in which there are three different groups of people who can be distinguished according to how many of four possible exemptions they are allowed to claim on their tax returns: zero for individuals in Group A, two for individuals in Group B, and four for individuals in Group C. These individuals are members of an exemption-maximizing but law-abiding society, so everyone in Group A claims zero exemptions, everyone in Group B claims exactly two, and everyone in group C claims exactly four. Now consider the following utterances as descriptions of this situation:

(35) a. No individual who was allowed to claim two exemptions claimed four.
b. No individual who was allowed to claim some exemptions claimed four.

(35a) has a reading in which it is true in this scenario, because the quantifier restriction is understood to pick out individuals who were allowed to to claim two exemptions, and not allowed to claim more than two exemptions, i.e. the ones in Group B. This is an upper bounded reading, but it occurs in a downward entailing context (and in the argument of a logical operator), which is a context in which scalar implicatures are suppressed. And indeed, (35b), in which the number word is replaced by the scalar quantifier *some*, is clearly false in this situation, because the quantifier is understood to range over all individuals who were allowed to claim exemptions, which includes the ones in Group C. The sentence does not have a reading in which the restriction ranges over individuals who were allowed to claim some exemptions but not allowed to claim all exemptions. This would exclude the individuals in Group C, and would make the sentence true.

Before turning to the analysis of modified numerals, let me say a few words about the relation between the de-Fregean quantificational meaning of a number word and the singular term meaning. For the purposes of this paper, it would be fine to assume that an unmodified numeral like *three* has only the de-Fregean denotation in (28), since this is the meaning that will play a role in accounting for the patterns of interpretations that we find Class A and Class B modified numerals. However, even though there is good reason to believe that the two-sided truth conditions that this meaning introduces are both a matter of semantic content and also a strong default, examples like the following show that there is still reason to think that lower-bounded, one-sided meanings are available as well:

(36) a. Kim took three classes, if not four.
b. No one who misses three questions on the exam will receive a drivers license.

If the first part of (36a) had only the two-sided interpretation in (29b), then the continuation in the second part should sound strange (Horn 1972). Similarly, the domain of the quantifier in (36b) is most naturally understood to be those people who miss at least three questions on the exam. There may be ways to accommodate these kinds of examples while maintaining a view in which numerals always introduce two-sided content (see Breheny 2008 for discussion), I will instead assume that numerals are ambiguous between a de-Fregean, type $\langle\langle d,t\rangle,t\rangle$ denotation, and a type d singular term denotation. This will simplify the analysis of modified numerals in the next section, and will accommodate data like (36): as we have seen, the de-Fregean meaning derives the two-sided interpretation of the first part of (36a), repeated in (37a), and the singular term meaning gives the one-sided, lower bounded truth conditions shown in (37b).

(37) a. $max\{n|\exists x[\textbf{took}(x)(\textbf{kim}) \wedge \textbf{classes}(x) \wedge \#(x) = n]\} = 3$
b. $\exists x[\textbf{took}(x)(\textbf{kim}) \wedge \textbf{classes}(x) \wedge \#(x) = 3]$

As it turns out, this ambiguity does not need to be stipulated, but follows from general type-shifting principles, if we assume that the de-Fregean meaning is basic. In particular, the singular term meaning can be derived from the de-Fregean meaning by successive application of Partee's (1987) *BE* and *iota* operations, defined in (38).

(38) a. *BE* = $\lambda \mathcal{Q}_{\langle\langle \alpha t\rangle ,t\rangle}\lambda x_{\alpha}.\mathcal{Q}(\lambda y_{\alpha}.y = x)$
b. *iota* = $\lambda P_{\langle \alpha ,t\rangle}.\iota x_{\alpha}[P(x)]$

BE maps a generalized quantifier to the property that is true of all the singletons in the denotation of the quantifier. Application of *BE* to the de-Fregean denotation of *three* derives the property of being a number equal to three, as shown by the following derivation:

(39) *BE*($[\![three]\!]$)

$$
\begin{aligned}
&= [\lambda \mathcal{Q}\lambda m.\mathcal{Q}(\lambda p.p = m)](\lambda P.max\{n \mid P(n)\} = 3) && \\
&= [\lambda m.[\lambda P.max\{n \mid P(n)\} = 3](\lambda p.p = m)] && (\lambda\text{-conversion})\\
&= \lambda m.max\{n \mid [\lambda p.p = m](n)\} = 3 && (\lambda\text{-conversion})\\
&= \lambda m.max\{n \mid n = m\} = 3 && (\lambda\text{-conversion})\\
&= \lambda m.m = 3 && (\text{substitution of equivalents})
\end{aligned}
$$

Application of *iota* to the result returns the unique number equal to three, which is of course three itself.[3]

It might seem counterintuitive to assume that it is the de-Fregean, quantificational meaning of the numeral that is basic, rather than the simpler singular term denotation. However, while the latter can be derived from the former using Partee's well-established type-shifting principles, the former cannot be so derived from the latter. Instead, it can only be derived through the addition of some meaning changing operation, such as exhaustification. These issues and their implications for the analysis of more complex sentences involving numerals are explored in Kennedy 2013a.

[3] In fact, if the grammar includes an operation like Chung and Ladusaw's Restrict, as I have assumed, then we don't actually need to map the de-Fregean meaning to a singular term in order to derive the one-sided interpretation of (36a); the $\langle d,t\rangle$ meaning in (39) is enough. Composition of this meaning with the type $\langle d,\langle e,t\rangle\rangle$ noun (or implicit cardinality predicate) meaning plus existential closure over the degree argument derives (i), which is equivalent to (37b).

(i) $\exists x\exists n[\textbf{took}(x)(\textbf{kim}) \wedge \textbf{classes}(x) \wedge \#(x) = n \wedge n = 3]$

3.2 *Modified numerals*

With this account of unmodified numerals in hand, we can now turn to the analysis of modified numerals, and the explanation for the patterns of data discussed in Sections 1 and 2. For Class A modifiers like *more than* and *fewer than*, I will assume with Nouwen (2010) that they combine with the singular term denotation of a numeral to give back a generalized quantifier over degrees with a comparative meaning, as in (40).

(40) a. $[\![more\ than]\!] = \lambda n \lambda P_{\langle d,t \rangle}.max\{n \mid P(n)\} > n$
b. $[\![fewer\ than]\!] = \lambda n \lambda P_{\langle d,t \rangle}.max\{n \mid P(n)\} < n$

For Class B modifiers, it is probably already obvious that I will need to provide them with denotations that are distinct from those proposed by Nouwen, because the de-Fregean semantics of unmodified numerals is identical to Nouwen's semantics for modified numerals with Class B maximizing modifiers like *at most*. This is arguably a good result, since one of challenges for Nouwen's analysis derived from the fact that *at most three* ended up producing meanings that were identical to the meanings of sentences containing bare *three*. As we saw, Nouwen used this result to invoke blocking principles and last-resort modalized interpretations to explain the properties of sentences containing *at most three*, but this approach led to the problems with modals that we saw in Section 2. Instead, I will propose that Class B modifiers are just like Class A modifiers except that they introduce partial orderings rather than total orderings:

(41) a. $[\![at\ least]\!] = \lambda m \lambda P_{\langle d,t \rangle}.max\{n \mid P(n)\} \geq m$
b. $[\![at\ most]\!] = \lambda m \lambda P_{\langle d,t \rangle}.max\{n \mid P(n)\} \leq m$

In fact, exactly this approach to Class B modifiers is considered by Nouwen himself, but rejected as inadequate.[4] I will explain and respond to Nouwen's criticisms below; first let me show how the approach accounts for the facts we considered in Sections 1 and 2.

3.2.1 *Uncertainty inferences*

My account of the ignorance/uncertainty inferences associated with Class B modifiers is of the same sort as the pragmatic accounts proposed by Büring (2008) and

[4] It is also used by Büring (2008), though only as a placeholder for the focus-based account that he eventually settles on. I will return to a discussion of focus-based analyses of superlative modifiers in section 4.

Cummins and Katsos (2010): they arise as conversational implicatures from the utterance of a sentence whose propositional content is less informative relative to potential alternatives, indicating uncertainty on the part of the speaker about whether those stronger alternatives hold. (See also Rett 2013, who adopts a similar line of reasoning for “measure phrase equatives” like *as many as three*.) The crucial question is what the alternatives are. An initial hypothesis is that the alternatives are propositions in which the the quantity that satisfies the scope clause of the modified numeral is related to some higher (or lower, for the negative modifiers) value than the one introduced by the numeral; e.g., the set of propositions in (42b) for (42b).

(42) a. A hexagon has at least three sides.
b. $\{max\{n \mid \exists x[\#(x) = n \wedge \textbf{sides}(x) \wedge \textbf{have}(x)(\textbf{hex})]\} \geq m \mid m > 3\}$

Geurts and Nouwen (2007) reject this view on the grounds that the alternative set for (42a) is equivalent to the alternative set of (43a), shown in (43b) — and more generally that the alternatives of *at least* n/*at most* n are equivalent to those of *more than* $n-1$/*fewer than* $n+1$, respectively — in which case (42a) and (43a) should have equal informativity (see also Nouwen 2008b). Yet only the former gives rise to ignorance inferences.

(43) a. A hexagon has more than two sides.
b. $\{max\{n \mid \exists x[\#(x) = n \wedge \textbf{sides}(x) \wedge \textbf{have}(x)(\textbf{hex})]\} > m \mid m > 2\}$

While I agree with Geurts and Nouwen that a pragmatic account of the presence vs. absence of ignorance inferences with Class A and B modifiers that relies on the idea that (42b) and (43b) represents the relevant alternatives to consider, I do not think that the reason is because (42a) and (43a) have the same propositional content. In particular, if Fox and Hackl (2007) are correct that measurement scales are dense, and the range of the measure function # includes degrees other than whole numbers (see the discussion of this point in footnote 1), then *at least* n picks out degrees that begin with (the number) n, but *more than* $n-1$ picks out those degrees ordered below n as well, up to but not including $n-1$. The fact that we think of (42a) and (43a) as equivalent is because we generally ignore degrees that represent quantities of non-whole objects, because we count in terms of wholes (cf. Salmon 1997). But examples like the following show that the semantics must have access to these values; if it did not, then *more than* n *but fewer than* $n+1$ would give rise to a contradiction:

(44) a. A blood alcohol calculator says that based on my weight I can have **more than three but fewer than four** beers in an hour and be below the legal limit. (http://community.thenest.com/cs/ks/forums/thread/50998406.aspx)

b. The dimensionality of a strange attractor is often not an integer. Rather, it is natural to define a *fractal dimension* for the attractor, which might be 2.3 for an attractor which occupies **more than two but fewer than three** dimensions in the 100-dimensional space. (Stuart A. Kauffman, *The Origins of Order: Self-Organization and Selection in Evolution*, p. 179.)

Unfortunately, we now seem to be in even worse shape: if the domain of # is dense, and *at least three* and *more than two* range over degrees in between those corresponding to whole numbers, then (42a) and (43a) are not only not logically equivalent, (42a) asymmetrically entails (43a). That makes the *at least* sentence more informative than the *more than* sentence, yet it is the former that gives rise to the uncertainty inference, not the latter.

Clearly, then, it cannot be the case that the alternatives defined in (42b) and (43b) are the ones that are relevant for the calculation of implicatures having to do with speaker certainty about quantity.[5] Intuitively, the reason why these alternatives are irrelevant for determining certainty about quantity is that every sentence involving a Class A/B modified numeral n has the same number of alternatives based on different values for n: an infinity of them. They can be organized into an entailment scale of relative strength, but they are all (in some sense) equally uninformative about actual quantity.

But there is a different alternative set that can be constructed for a sentence containing a de-Fregean (modified or unmodified) numeral that gives us exactly the result that we want: the set of propositions that hold the degree contributed by the numeral constant, but vary the relation that holds between this degree and the maximal degree that satisfies the degree property provided by the scope of the quantifier. In the case of (42a), which has the content in (45a), the alternatives are the propositions in (45b-c) — which are just the propositions expressed by the corresponding sentence with unmodified *three* and the one with *more than three*.

(45) a. $max\{n \mid \exists x[\#(x) = n \wedge \textbf{sides}(x) \wedge \textbf{have}(x)(\textbf{hex})]\} \geq 3$ (= CLASS B)
b. $max\{n \mid \exists x[\#(x) = n \wedge \textbf{sides}(x) \wedge \textbf{have}(x)(\textbf{hex})]\} = 3$ (= UNMODIFIED)
c. $max\{n \mid \exists x[\#(x) = n \wedge \textbf{sides}(x) \wedge \textbf{have}(x)(\textbf{hex})]\} > 3$ (= CLASS A)

(45a) is entailed by both (45b) and (45c), but not vice versa, a fact that holds generally for Class B vs. unmodified and Class A modified numerals. An assertion of the *at least* sentence with the content in (45a) and the extra complexity introduced

[5]These alternatives may, however, be precisely the ones we want for the calculation of scalar implicature. Fox and Hackl (2007) show how the alternative set in (43b), combined with scalar density, explains the fact that *more than* n does not implicate *exactly* $n+1$. The absence of a similar scalar implicature in (42a) (from *at least* n to *exactly* n does not follow from Fox and Hackl's account, but instead follows from the account of uncertainty inferences proposed here.

by the modifier implicates that the speaker is not in an epistemic position to assert the sentences with the stronger meanings in (45b-c): if she knows for sure that the number of sides of a hexagon is three, she should assert the syntactically simpler bare numeral sentence; if she knows for sure that the number of sides is not three but is also not less than three, then she should assert the *more than* sentence. Since she chooses the option that is consistent with each of these stronger but mutually exclusive alternatives, she must not be in a position to assert either of them. But precisely because the stronger options are mutually exclusive, we do not draw the additional scalar inference that one or the other of the stronger propositions is false. Doing so would give us a meaning equivalent to the other stronger version — *at least three and not three* equals *more than three*, and *at least three and not more than three* equals *three* — which would then conflict with our initial assumptions about the speaker's knowledge state, namely that she is not in a position to assert either of these. The end result is one in which both of the stronger alternatives must be "live options," so an utterance of *at least three* is, in effect, equivalent to an utterance of *three or more than three*, as has already been pointed out by Büring (2008) and Cummins and Katsos (2010), though on the account sketched here, the disjunction is not in any way part of the meaning of *at least*. The same kind of reasoning can be extended to *at most* sentences, where the alternatives are the bare numeral and the *fewer than* variant.

3.2.2 Interactions with modals

Let us now look at the interactions of Class B modifiers with modals. First, it should be clear that we have no problem with epistemic modals: both bare and modified numerals are degree quantifiers, and (for whatever reason) cannot take scope over epistemic modals. This was a bit of a puzzle for Nouwen because his account of uncertainty inferences crucially relied on inserting an epistemic modal inside the scope of a Class B modifier. Since the current account does not rely on the presence of an epistemic modal to generate uncertainty inferences, this problem disappears.

The real test, then, is the account of interactions with root modals. Let us go through the crucial facts step-by-step, starting with universal modals and minimizing Class B modifiers. Recall that in Nouwen's analysis, (46) is incorrectly predicted to entail that registration in more than three classes is not allowed, regardless of whether the modified numeral takes scope above or below the modal, as shown by the logical forms in (47a-b). (Since my goal now is to describe the possible interpretations of various sentences, rather than to illustrate the role of blocking in Nouwen's analysis, I will use "*" uniformly in the following examples to indicate that the truth conditions indicated by a particular logical representation are unavailable.)

(46) You are required to register for minimally/at least three classes.

(47) a. * $\Box[min\{n \mid \exists!x[\textbf{reg}(x)(\textbf{you}) \wedge \textbf{classes}(x) \wedge \#(x) = n]\} = 3]$

b. * $min\{n \mid \Box[\exists!x[\textbf{reg}(x)(\textbf{you}) \wedge \textbf{classes}(x) \wedge \#(x) = n]]\} = 3$

(47a) is ruled out in Nouwen's analysis by blocking, and the actual meaning of (46) is derived by converting the universal modal in (47b) to an existential modal.

In contrast, the denotation for *at least* in (41a) gives the correct truth conditions regardless of whether the modified numeral scopes above or below the modal. The two potential interpretations of (46) are shown in (48).

(48) a. $\Box[max\{n \mid \exists x[\textbf{reg}(x)(\textbf{you}) \wedge \textbf{classes}(x) \wedge \#(x) = n]\} \geq 3]$

b. $max\{n \mid \Box[\exists x[\textbf{reg}(x)(\textbf{you}) \wedge \textbf{classes}(x) \wedge \#(x) = n]]\} \geq 3$

(48a) is true just in case every deontically accessible world is one that involves registration in at least three classes; (48b) is true just in case the maximum n such that in every deontically accessible world there is registration in at least n classes is at least three. These are equivalent truth conditions, and correspond to the intuitively correct meaning of the sentence. (They give rise to distinct implicatures, though; see Büring 2008 for discussion.)

Now consider maximizing Class B modifiers with a universal deontic modal:

(49) You are required to register for at most three classes.

Here the problem for Nouwen's analysis was that the predicted meanings for both scope relations are incorrect: when the numeral scopes below the modal, the meaning should be that registration in exactly three classes is required; when it scopes above the modal, the meaning should be that registration in at least three classes is required:

(50) a. * $\Box[max\{n \mid \exists x[\textbf{reg}(x)(\textbf{you}) \wedge \textbf{classes}(x) \wedge \#(x) = n]\} = 3]$

b. * $max\{n \mid \Box[\exists x[\textbf{reg}(x)(\textbf{you}) \wedge \textbf{classes}(x) \wedge \#(x) = n]]\} = 3$

In other words, because Nouwen's semantics for *at most* is equivalent to the semantics for bare numerals on the de-Fregean analysis, the result is truth conditions that are equivalent for the corresponding sentence with an unmodified numeral.

In contrast, the semantics for *at most* in (41b) assigns the denotation in (51a) to (49) when the numeral scopes below the modal, and the one in (51b) when it scopes above the modal.

(51) a. $\Box[max\{n \mid \exists x[\textbf{reg}(x)(\textbf{you}) \wedge \textbf{classes}(x) \wedge \#(x) = n]\} \leq 3]$

b. $max\{n \mid \Box[\exists x[\textbf{reg}(x)(\textbf{you}) \wedge \textbf{classes}(x) \wedge \#(x) = n]]\} \leq 3$

(51a) says that every deontically accessible world is such that there is registration in three or fewer classes. This is the most salient interpretation of (49). (51b) says that the maximal n such that in every accessible world there is registration in at least n classes is less than or equal to three. This reading allows for enrollment in more than three classes, and indeed this seems to be a possible interpretation of this sentence:

(52) You are required to register for at most four classes during your first three years in the program, but I can't remember exactly how many. And of course registration in more is always an option.

Now consider minimizing Class B modifiers and existential modals:

(53) You are allowed to register for at least three classes.

In Nouwen's analysis, when the numeral takes scope below the modal, as in (54a), the predicted truth conditions are that registration in exactly three classes is allowed:

(54) a. $\Diamond[min\{n \mid \exists!x[\mathbf{reg}(x)(\mathbf{you}) \wedge \mathbf{classes}(x) \wedge \#(x) = n]\} = 3]$
b. $*\ min\{n \mid \Diamond[\exists!x[\mathbf{reg}(x)(\mathbf{you}) \wedge \mathbf{classes}(x) \wedge \#(x) = n]]\} = 3$

This is indeed a reading of (53), but it should be blocked in Nouwen's analysis, because it is equivalent to one of the readings that we get with the bare numeral. When the numeral takes scope over the modal, as in (54b), the predicted reading is that the minimum n such that there is a deontically accessible world with exactly n registered-for classes is three, which correctly lets in registration in more than three classes, but incorrectly rules out registration in one or two classes.

The truth conditions derived by the analysis I have proposed here are shown in (55a-b), which again represent modal over numeral scope and numeral over modal scope, respectively.

(55) a. $\Diamond[max\{n \mid \exists x[\mathbf{reg}(x)(\mathbf{you}) \wedge \mathbf{classes}(x) \wedge \#(x) = n]\} \geq 3]$
b. $max\{n \mid \Diamond[\exists x[\mathbf{reg}(x)(\mathbf{you}) \wedge \mathbf{classes}(x) \wedge \#(x) = n]]\} \geq 3$

(55a) is quite weak: it says merely that there is a deontically accessible world in which the maximum number of registered-for classes is three or more. (55b) is a bit stronger: it says that the maximum number of allowable, registered-for classes is three or more, which does not entail that it is forbidden to register in one or two classes. Thus we resolve the problem that Nouwen's analysis ran into.

Finally, let us turn to maximizing Class B modifiers and existential modals, which represent the one case that looks problematic for the analysis of Class B modifiers that I have proposed here:

(56) You are allowed to register for at most three classes.

Recall that in Nouwen's analysis, the truth conditions derived for this sentence when the numeral scopes below the modal, shown in (57a), are identical to the weak reading of the bare numeral sentence, and merely say that registration in exactly three classes is permissible.

(57) a. $\Diamond[max\{n \mid \exists x[\textbf{reg}(x)(\textbf{you}) \wedge \textbf{classes}(x) \wedge \#(x) = n]\} = 3]$
b. $max\{n \mid \Diamond[\exists x[\textbf{reg}(x)(\textbf{you}) \wedge \textbf{classes}(x) \wedge \#(x) = n]]\} = 3$

In Nouwen's analysis, this reading is ruled out by blocking, leaving only the wide-scope truth conditions in (57b), which say that the maximal n such that there is a deontically accessible world with registration in at least n classes is three. So (55) is predicted to have only this strong interpretation.

The analysis that I have proposed here also derives the strong interpretation of (56) by scoping the numeral above the modal. The problem is that it also derives an apparently unavailable weak reading for (56) when the numeral takes scope below the modal; it is for this reason that Nouwen (2010) rejects the semantics of Class B modifiers that I presented in (41) as inadequate, as I mentioned above. The two predicted interpretations of (56) are shown in (58a-b) in the usual order:

(58) a. $\Diamond[max\{n \mid \exists x[\textbf{reg}(x)(\textbf{you}) \wedge \textbf{classes}(x) \wedge \#(x) = n]\} \leq 3]$
b. $max\{n \mid \Diamond[\exists x[\textbf{reg}(x)(\textbf{you}) \wedge \textbf{classes}(x) \wedge \#(x) = n]]\} \leq 3$

First consider (58b): this says that the maximal n such that there is a deontically accessible world in which you register for at least n classes is less than or equal to three. This rules out registration in more than three classes, but it does not actually entail that registration in three classes is allowed, and in this way it differs crucially from the truth conditions that Nouwen's analysis delivers, which are instead equivalent to the (strong) meaning that we derive for bare numerals on the de-Fregean analysis I have proposed here. This is actually a good result, I would argue, since *"allow ... at most three"* differs from *"allow ... three"* in being consistent with uncertainty about the actual upper bound of allowed classes:

(59) a. Students are allowed to drop at most three of their classes, but I don't know the exact number. Maybe they can only drop two.
b. # Students are allowed to drop three of their classes, but I don't know the exact number. Maybe they can only drop two.

If the first clause in (59a) really had a meaning parallel to (57a), which is the one that Nouwen's analysis would assign to it, then the fact that it is possible to follow it up with a claim of uncertainty about the upper bound of allowed course drops would be surprising. On the other hand, the fact that such a follow up is unacceptable in (59b) shows that this is precisely the meaning that we want for the unmodified numeral, which in turn provides more support for the de-Fregean analysis.

Now let us consider the narrow scope interpretation of *at most three* in (58a). Nouwen (2010) observes that if such readings are in general available, then it should be possible to use an *at most* sentence in a permission claim without introducing an upper bound (see also Geurts and Nouwen 2007; Büring 2008). This is the case with both the Class A modifier *fewer than* and with the bare numeral, as shown by the acceptability of the second sentence in (60).

(60) John is allowed to bring (fewer than) ten friends. But more is fine too.

But, Nouwen argues, this is not the case with the Class B modifier *at most*, pointing to the infelicity of the continuation in (61).

(61) John is allowed to bring at most ten friends. #But more is fine too.

(65a-c) show the relevant interpretations, and provide a summary of the data: interpretations in which the numeral scopes under the modal, which do not place an upper-bound on the permitted quantity, are available for bare numerals and Class A modified numerals, but appear to be unavailable for Class B modified numerals.

(62)
a. *allowed > three*
$\Diamond[max\{n \mid \exists x[\textbf{bring}(x)(\textbf{j}) \wedge \textbf{friends}(x) \wedge \#(x) = n]\} = 10]$
b. *allowed > fewer than three*
$\Diamond[max\{n \mid \exists x[\textbf{bring}(x)(\textbf{j}) \wedge \textbf{friends}(x) \wedge \#(x) = n]\} < 10]$
c. $\otimes$ *allowed > at most three*
$\Diamond[max\{n \mid \exists x[\textbf{bring}(x)(\textbf{j}) \wedge \textbf{friends}(x) \wedge \#(x) = n]\} \leq 10]$

Nouwen concludes from this pattern that (41b) is not the correct semantics for *at most* in the first place. In order to preserve my analysis, then, I need to find some other explanation for the apparent absence of narrow scope readings of *at most* under existential deontic modals.

In fact, I think that the apparent absence of a narrow scope reading of *at most* in (61) is not a grammatical or semantic fact, and instead follows from pragmatic reasoning about permissions and the uncertainty inferences that we already know Class B modifiers give rise to. Note that, just as in non-modalized sentences, the modalized sentence with a bare numeral in (65a) and the one with the Class A modified numeral in (65b) asymmetrically entail the sentence with the Class B modified numeral in (65c). As a result, an utterance of a sentence that conveys (65c) should implicate that the speaker does not know whether (65a-b) hold; i.e., that the speaker is uncertain whether bringing exactly ten friends is permitted and is uncertain whether bringing fewer than ten friends is permitted, and so is uncertain about exactly what number of friends between one and ten she can bring. At the same time, the continuation *"But more is fine, too."* indicates that the speaker is certain that bringing eleven or more friends is permitted. But in the typical case, if a speaker is uncertain about whether bringing a smaller number of friends is permitted, then she ought to also be uncertain about whether bringing a larger number of friends is permitted, since bringing more involves a greater imposition than bringing less. Or, to put it another way, if the speaker is certain that it is acceptable to bring eleven or more friends, she should be at least as certain that it is acceptable to bring one to ten friends, since bringing fewer friends involves less of an imposition than bringing more. The infelicity of (61), then, can be explained in terms of the incompatibility of the uncertainty inferences of the *at most* sentence and the certainty inferences of the *more is fine* sentence.

If this reasoning is correct, then I predict that a narrow scope interpretation of *at most* relative to an existential deontic modal should be acceptable as long as the discourse is consistent with simultaneous uncertainty about whether lower values are permitted and certainty about whether higher values are permitted. The following scenario illustrates such a discourse:

(63) A: Next quarter I really want to focus on writing my qualifying papers, and I don't know exactly how much coursework I can manage at the same time, but I'm certain that I can't handle more than three classes. I know that the normal load is five classes. Am I allowed to register for at most three?

B: Yes, you're allowed to register for at most three classes.

Here the narrow scope interpretation of *at most* relative to *allowed* is the only one that makes sense, since Speaker A has indicated his awareness of the fact that the wide scope reading is false (it entails that taking more than three classes is not allowed). But the context is nevertheless one in which Speaker A is uncertain both about how many classes he will take, and about whether registration in one, two, or three courses (i.e., fewer than the "normal load") is consistent with the rules. These are precisely the conditions that we need to license the narrow scope reading of *at most*.

4 Concluding remarks

In this paper, I have presented a "de-Fregean" semantics for modified and unmodified numerals as generalized quantifiers over degrees in which bare numerals, Class A modified numerals and Class B modified numerals all have essentially the same semantic analysis, differing only in the kind of ordering relation they introduce over the unique (maximal) degree that satisfies their scope. (64) illustrates the basic pattern for bare *nine*, Class A modified *more than nine*, and Class B modified *at least nine*.

(64) a. ⟦nine⟧ = $\lambda P_{\langle d,t\rangle}.max\{n \mid P(n)\} = 9$
b. ⟦more than nine⟧ = $\lambda P_{\langle d,t\rangle}.max\{n \mid P(n)\} > 9$
c. ⟦at least nine⟧ = $\lambda P_{\langle d,t\rangle}.max\{n \mid P(n)\} \geq 9$

I have shown that this account can derive the uncertainty inferences associated with Class B modifiers as an implicature that arises from the fact that sentences with bare numerals and Class A modified numerals asymmetrically entail the corresponding sentences with Class B modified numerals, and I have shown that this analysis correctly captures the truth conditions of sentences involving scopal interactions between numerals and modals. There are a number of respects in which the proposals I have made are incomplete, however.

First, there are individual differences between the modifiers in the Class A and Class B groups having to do with, for example, whether they are compatible with a single degree satisfying their scope, or whether they require an interval (Nouwen 2008a; Schwarz et al. 2012; Rett 2013). A more fine-grained analysis of the lexical semantics of the various modifiers and the way their uses as numeral modifiers relate to their other uses as comparative or locative morphemes is necessary to provide a full account of these differences.

Second, I have said nothing about how the degree semantics of modified and unmodified numerals interacts with quantification over the individual argument of the nominal, and in particular with the collective/distributive distinction (see Brasoveanu 2013 for recent discussion), other than to stipulate that the latter is bound by a default existential quantifier. These issues are taken up in Kennedy 2013a.

Finally, and most significantly, the proposals I have made here are surely insufficient as a full semantic account of (at least) the superlative modifiers *at least* and *at most* (and probably the adverbials *minimally* and *maximally* as well). The denotations that I gave for these modifiers in section 3.2 treat them as expressions that map degrees (in particular, singular term numeral denotations) to generalized quantifiers over degrees, which leads to the prediction that they should combine only with degree-denoting expressions. But this prediction is wrong: it is a well-known feature of these expressions both that they can combine with a range of different categories, as shown in (65b-c). (I use *at most* for illustration, but similar examples can be constructed with *at least*.)

(65) a. We should invite at most three linguists, not two.
b. We should invite at most Kim, Lee and Pat, not Mo as well.

c. We should invite at most some linguists, not some philosophers as well.

They can, moreover, be separated from the expression with which they are understood to associate:

(66) a. We should at most invite three linguists.
b. We should at most invite Kim, Lee and Pat.
c. We should at most invite some linguists.

(67) a. We should invite three linguists at most.
b. We should invite Kim, Lee and Pat at most.
c. We should invite some linguists at most.

This latter syntactic fact is reflected in the semantics by focus-sensitivity: in both split and non-split forms, focus determines what is understood to be the associate of the modifier.

(68) a. We should at most invite THREE linguists, not four/??three philosophers.
b. We should at most invite three LINGUISTS, not ??four/three philosophers.

(69) a. We should invite at most THREE linguists, not FOUR/??three PHILOSOPHERS.
b. We should invite at most three LINGUISTS, not ??FOUR/three PHILOSOPHERS.

These kinds of facts have led a number of authors to develop focus-based accounts of these modifiers (see e.g. Krifka 1998; Geurts and Nouwen 2007; Cohen and Krifka 2010), with recent work by Coppock and Brochhagen (to appear) geared specifically towards showing how this kind of approach, together with an inquisitive semantics, can capture the Class A/Class B distinction. Given that such an approach is likely to be independently necessary in order to account for focus-sensitivity, cross-categoriality, and split variants of modified numerals, it is worth asking whether this will ultimately make the degree theoretic, de-Fregean analysis I have proposed here superfluous.

Although a full answer to this question goes beyond the scope of this paper, my guess that both kinds of analyses will ultimately be necessary, and that a fully comprehensive treatment of modified and unmodified numerals will involve integrating the focus-sensitive and degree theoretic approaches. The latter provides us with a compositional framework that reflects the growing evidence that modified and unmodified numerals are syntactic and semantic constituents that saturate a quantity position inside the nominal projection and can take scope independently of the nominal with which they compose on the surface (see e.g. Hackl 2000; Heim

2000; Takahashi 2006; Nouwen 2008b; Kennedy and Stanley 2009; Nouwen 2010; Kennedy 2013b).

It also provides us with a simple account of minimal pairs like the following:

(70) a. # A hexagon has no more than ten sides.
b. A hexagon does not have more than ten sides.

(71) a. # A hexagon has no fewer than four sides.
b. A hexagon does not have fewer than four sides.

In a fully compositional semantics of comparatives, *more/fewer than* are properly analyzed not as introducing the simple ordering relations $>$ and $<$, but as introducing a requirement for a positive difference in degree (von Stechow 1984; Schwarzschild 2005, see e.g.). The function of the negative differential expression *no* is to say that there is no positive difference (Alrenga and Kennedy in press), so *no more than* is consistent with either $=$ or $<$, and *no fewer than* is consistent with either $=$ or $>$. The result is that *no more/fewer than* have the same ordering entailments as *at most/at least*, so it is no surprise, from a degree-theoretic perspective, that they show the Class B uncertainty pattern.[6]

Given these considerations, it seems that even a focus-based analysis of superlative modifiers needs to hypothesize such expressions to take on meanings that allow them to combine with numerals to derive generalized quantifiers over degrees. And indeed, this is the approach taken by Coppock and Brochhagen (to appear), who show that the basic meanings they assume for *at least* and *at most* as propositional modifiers can be shifted into meanings that combine with a numeral, generating a generalized quantifier over degrees which introduces truth conditions that are essentially equivalent to what we get on the analysis of *at least/most* n that I have proposed here, but are based in a more sophisticated analysis of how the conventional meaning of the modifier introduces constraints on how the assertion relates to the question under discussion. This part of the focus-based analysis (or at least, of Coppock and Brochhagen's version of it) is what gives rise to the uncertainty inferences of Class B modifiers, and may ultimately turn out to provide a more robust account of this phenomenon than the fully pragmatic one I have given here.

[6]What is a surprise is that unlike *at most/least* and other Class B modifiers, they can can give rise to scalar bounding inferences, so that e.g. *no more than ten* is often understood to convey *ten*. (This is not the reason for the unacceptability of (70a) and (71a), however: as we saw at the very beginning of the paper, it is perfectly acceptable to falsely assert that a hexagon has ten or four sides.) Nouwen (2008b) concludes from this that *at least/most* and other Class B modifiers cannot mean the same thing as *no fewer than/no more than*, but the fact that *no more/fewer than* also introduce uncertainty inferences (a fact that Nouwen does not mention) shows that the situation is more complicated. I do not know what is going on here, though I suspect that the extra complexity of *no more/fewer than* — differential *no* is itself a degree quantifier (Alrenga and Kennedy in press) — is significant.

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