Chapter 3

Properties of Relations

3.1 Reflexivity, symmetry, transitivity, and connectedness

Certain properties of binary relations are so frequently encountered that it is useful to have names for them. The properties we shall consider are *reflexivity*, *symmetry*, *transitivity*, and *connectedness*. All these apply only to relations *in* a set, i.e., in $A \times A$ for example, not to relations from A to B, where $B \neq A$.

Reflexivity

Given a set A and a relation R in A, R is *reflexive* if and only if all the ordered pairs of the form $\langle x, x \rangle$ are in R for every x in A.

As an example, take the set $A = \{1, 2, 3\}$ and the relation $R_1 = \{\langle 1, 1 \rangle, \langle 2, 2 \rangle, \langle 3, 3 \rangle, \langle 3, 1 \rangle\}$ in A. R_1 is reflexive because it contains the ordered pairs $\langle 1, 1 \rangle, \langle 2, 2 \rangle$, and $\langle 3, 3 \rangle$. The relation $R_2 = \{\langle 1, 1 \rangle, \langle 2, 2 \rangle\}$ is nonreflexive since it lacks the ordered pair $\langle 3, 3 \rangle$ and thus fails to meet the definitional requirement that it contains the ordered pair $\langle x, x \rangle$ *for every* x *in* A. Another way to state the definition of reflexivity is to say that a relation R in A is reflexive if and only if id_A, the identity relation in A, is a subset of R. The relation 'has the same birthday as' in the set of human beings is reflexive.

A relation which fails to be reflexive is called nonreflexive, but if it contains *no* ordered pair $\langle x, x \rangle$ with identical first and second members, it is said to be *irreflexive*. $R_3 = \{\langle 1, 2 \rangle, \langle 3, 2 \rangle\}$ is an example of an irreflexive relation in A. Irreflexivity is a stronger condition than nonreflexivity since every irreflexive relation is nonreflexive but not conversely. The relation 'is taller than' in the set of human beings is irreflexive (therefore also nonreflexive), while the relation 'is a financial supporter of' is nonreflexive (but not irreflexive, since some people are financially self-supporting). Note that a relation R in A is nonreflexive if and only if $id_A \not\subseteq R$; it is irreflexive if and only if $R \cap id_A = \emptyset$.

Symmetry

Given a set A and a binary relation R in A, R is *symmetric* if and only if for every ordered pair $\langle x, y\rangle$ in R, the pair $\langle y, x\rangle$ is also in R. It is important to note that this definition does not require every ordered pair of $A \times A$ to be in R. Rather for a relation R to be symmetric it must always be the case that *if* an ordered pair is in R, *then* the pair with the members reversed is also in R.

Here are some examples of symmetric relations in $\{1,2,3\}$:

(3–1) $\{\langle 1,2\rangle, \langle 2,1\rangle, \langle 3,2\rangle, \langle 2,3\rangle\}$
$\{\langle 1,3\rangle, \langle 3,1\rangle\}$
$\{\langle 2,2\rangle\}$

$\{\langle 2,2\rangle\}$ is a symmetric relation because for every ordered pair in it, i.e., $\langle 2,2\rangle$, it is true that the ordered pair with the first and second members reversed, i.e., $\langle 2,2\rangle$, is in the relation. Another example of a symmetric relation is 'is a cousin of' on the set of human beings. If for some $\langle x, y\rangle$ in R, the pair $\langle y, x\rangle$ is not in R then R is *nonsymmetric*. The relation 'is a sister of' on the set of human beings is nonsymmetric (since the second member may be male; it is, however, a symmetric relation defined on the set of human females).

The following relations in $\{1,2,3\}$ are nonsymmetric:

(3–2) $\{\langle 2,3\rangle, \langle 1,2\rangle\}$
$\{\langle 3,3\rangle, \langle 1,3\rangle\}$
$\{\langle 1,2\rangle, \langle 2,1\rangle, \langle 2,2\rangle, \langle 1,1\rangle, \langle 2,3\rangle\}$

If it is *never* the case that for any $\langle x, y\rangle$ in R, the pair $\langle y, x\rangle$ is in R, then the relation is called *asymmetric*. The relation 'is older than' is asymmetric on the set of human beings. Note that an asymmetric relation must be irreflexive (because nothing in the asymmetry definition requires x and y to be distinct). The following are examples of asymmetric relations in $\{1,2,3\}$:

(3–3) $\{\langle 2,3\rangle, \langle 1,2\rangle\}$
$\{\langle 1,3\rangle, \langle 2,3\rangle, \langle 1,2\rangle\}$
$\{\langle 3,2\rangle\}$

A relation is *anti-symmetric* if whenever both $\langle x, y\rangle$ and $\langle y, x\rangle$ are in R, then $x = y$. This definition says only that *if* both $\langle x, y\rangle$ and $\langle y, x\rangle$ are in R, then x and y are identical; it does not require $\langle x, x\rangle \in R$ for all $x \in A$. In other words, the relation need not be reflexive in order to be anti-symmetric.

The following relations in $\{1,2,3\}$ are anti-symmetric:

(3–4) $\{\langle 2,3\rangle, \langle 1,1\rangle\}$
$\{\langle 1,1\rangle, \langle 2,2\rangle\}$
$\{\langle 1,2\rangle, \langle 2,3\rangle\}$

Transitivity

A relation R is *transitive* if and only if for all ordered pairs $\langle x, y\rangle$ and $\langle y, z\rangle$ in R, the pair $\langle x, z\rangle$ is also in R.

Because there is no necessity for x, y, and z all to be distinct, the following relation meets the definition of transitivity,

(3–5) $\{\langle 2,2\rangle\}$

where $x = y = z = 2$.

The relation given in (3-6) is not transitive,

(3–6) $\{\langle 2,3\rangle, \langle 3,2\rangle, \langle 2,2\rangle\}$

because $\langle 3,2\rangle$ and $\langle 2,3\rangle$ are members, but $\langle 3,3\rangle$ is not.

Here are some more examples of transitive relations:

(3–7) $\{\langle 1,2\rangle, \langle 2,3\rangle, \langle 1,3\rangle\}$
$\{\langle 1,2\rangle, \langle 2,1\rangle, \langle 1,1\rangle, \langle 2,2\rangle\}$
$\{\langle 1,2\rangle, \langle 2,3\rangle, \langle 1,3\rangle, \langle 3,2\rangle, \langle 2,1\rangle, \langle 3,1\rangle, \langle 1,1\rangle, \langle 2,2\rangle, \langle 3,3\rangle\}$

The relation 'is an ancestor of' is transitive in the set of human beings. If a relation fails to meet the definition of transitivity, it is *nontransitive*. If for no pairs $\langle x, y\rangle$ and $\langle y, z\rangle$ in R, the ordered pair $\langle x, z\rangle$ is in R, then the relation is *intransitive*. For example, the relation 'is the mother of' in the set of human beings is intransitive.

Relation (3-6) is nontransitive, as are the following two:

(3–8) $\{\langle 1,2\rangle,\langle 2,3\rangle\}$
$\{\langle 1,2\rangle,\langle 2,3\rangle,\langle 1,3\rangle,\langle 3,1\rangle\}$

The first of these relations is also intransitive, as are the following relations:

(3–9) $\{\langle 3,1\rangle,\langle 1,2\rangle,\langle 2,3\rangle\}$
$\{\langle 3,2\rangle,\langle 1,3\rangle\}$

Connectedness

A relation R in A is *connected* (or *connex*) if and only if for every two *distinct* elements x and y in A, $\langle x,y\rangle \in R$ or $\langle y,x\rangle \in R$ (or both).

Note that the definition of connectedness refers, as does the definition of reflexivity, to all the members of the set A. Further, the pairs $\langle x,y\rangle$ and $\langle y,x\rangle$ mentioned in the definition are explicitly specified as containing nonidentical first and second members. Pairs of the form $\langle x,x\rangle$ are not prohibited in a connected relation, but they are irrelevant in determining connectedness.

The following relations in $\{1,2,3\}$ are connected:

(3–10) $\{\langle 1,2\rangle,\langle 3,1\rangle,\langle 3,2\rangle\}$
$\{\langle 1,1\rangle,\langle 2,3\rangle,\langle 1,2\rangle,\langle 3,1\rangle,\langle 2,2\rangle\}$

The following relations in $\{1,2,3\}$, which fail the definition, are nonconnected:

(3–11) $\{\langle 1,2\rangle,\langle 2,3\rangle\}$
$\{\langle 1,3\rangle,\langle 3,1\rangle,\langle 2,2\rangle,\langle 3,2\rangle\}$

It may be useful at this point to give some examples of relations specified by predicates and to consider their properties of reflexivity, symmetry, transitivity, and connectedness.

(3–12) *Example*: R_f is the relation 'is father of' in the set H of all human beings. R_f is irreflexive (no one is his own father); asymmetric (if x is y's father, then it is never true that y is x's father); intransitive (if x is y's father and y is z's father, then x is z's grandfather but not z's father); and nonconnected (there are distinct individuals x and y in H such that neither 'x is the father of y' nor 'y is the father of x' is true).

(3–13) *Example*: R is the relation 'greater than' defined in the set $Z = \{1,2,3,4,\ldots\}$ of all the positive integers. Z contains an infinite number of members and so does R, but we are able to determine the relevant properties of R from our knowledge of the properties of numbers in general. R is irreflexive (no number is greater than itself); asymmetric (if $x > y$, then $y \not> x$; transitive (if $x > y$ and $y > z$, then $x > z$), and connected (for every distinct pair of integers x and y, either $x > y$ or $y > x$.

(3–14) *Example*: R_a is the relation defined by 'x is the same age as y,' in the set H of all living human beings. R_a is reflexive (everyone is the same age as himself or herself); symmetric (if x is the same age as y, then y is the same age as x); transitive (if x and y are the same age and so are y and z, then x is the same age as z); and nonconnected (there are distinct individuals in H who are not of the same age).

3.2 Diagrams of relations

It may be helpful in assimilating the notions of reflexivity, symmetry and transitivity to represent them in relational diagrams. The members of the relevant set are represented by labeled points (the particular spatial arrangement of them is irrelevant). If x is related to y, i.e. $\langle x,y\rangle \in R$, an arrow connects the corresponding points. For example,

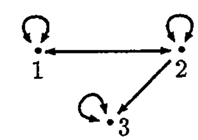

Figure 3–1: Relational diagram

Figure 3-1 represents the relation

$$R = \{\langle 1,2\rangle,\langle 2,1\rangle,\langle 2,2\rangle,\langle 1,1\rangle,\langle 2,3\rangle,\langle 3,3\rangle\}$$

It is apparent from the diagram that the relation is reflexive, since every point bears a loop. The relation is nonsymmetric since 3 is not related to 2 whereas 2 is related to 3. It cannot be called asymmetric or antisymmetric, however, since 1 is related to 2 and 2 is related to 1. It is nontransitive since 1 is related to 2 and 2 is related to 3, but there is no direct arrow from 1 to 3. The relation cannot be intransitive because of the presence of pairs such as $\langle 1, 1\rangle$.

If a relation is connected, every pair of distinct points in its diagram will be directly joined by an arrow. We see that R is not connected since there is no direct connection between 1 and 3 in Figure 3-1.

3.3 Properties of inverses and complements

Given that a relation R has certain properties of reflexivity, symmetry, transitivity or connectedness, one can often make general statements about the question whether these properties are preserved when the inverse R^{-1} or complement R' of that relation is formed.

For example, take a reflexive relation R in A. By the definition of reflexive relations, for every $x \in A$, $\langle x, x\rangle \in R$. Since R^{-1} has all the ordered pairs of R, but with the first and second members reversed, then every pair $\langle x, x\rangle$ is also in R^{-1}. So the inverse of R is reflexive also. The complement R' contains all ordered pairs in $A \times A$ that are not in R. Since R contains every pair of the form $\langle x, x\rangle$ for any $x \in A$, R' contains none of them. The complement relation is therefore irreflexive.

As another example, take a symmetric relation R in A. Does its complement have this property? Let's assume that the complement R' is not symmetric, and see what we can derive from that assumption. If R' is not symmetric, then there is some $\langle x, y\rangle \in R'$ such that $\langle y, x\rangle \notin R'$, by the definition of a nonsymmetric relation. Since $\langle y, x\rangle \notin R'$, $\langle y, x\rangle$ must be in the complement of R', which is R itself. Because R is symmetric, $\langle x, y\rangle$ must also be in R. But one and the same ordered pair $\langle x, y\rangle$ cannot be both in R *and* in its complement R', so the assumption that the complement R' is not symmetric leads to an absurd conclusion. That means that the assumption cannot be true and the complement R' must be symmetric after all. If R is a symmetric relation in A, then the complement R' is symmetric and vice versa (the latter follows from essentially the same reasoning with R' substituted for R). This mode of reasoning is an instance of what is called a *reductio ad absurdum* proof in logic. It is characterized by making an assumption which leads to a necessarily false conclusion; you may then conclude that

the negation of that assumption is true. In Chapter 6 we will introduce rules of inference which will allow such arguments to be made completely precise.

For sake of easy reference the table in Figure 3-2 presents a summary of properties of relations and those of their inverses and complements. These can all be proved on the basis of the definitions of the concepts and the laws of set theory. Since we have not yet introduced a formal notion of proof, we will not offer proofs here, but it is a good exercise to convince yourself of the facts by trying out a few examples, reasoning informally along the lines illustrated above.

R (not $\emptyset$)	R^{-1}	R'
reflexive	reflexive	irreflexive
irreflexive	irreflexive	reflexive
symmetric	symmetric ($R^{-1} = R$)	symmetric
asymmetric	asymmetric	nonsymmetric
anti-symmetric	anti-symmetric	depends on R
transitive	transitive	depends on R
intransitive	intransitive	depends on R
connected	connected	depends on R

Figure 3–2: Preservation of properties of a relation in its inverse and its complement

3.4 Equivalence relations and partitions

An especially important class of relations are the *equivalence relations*. They are relations which are reflexive, symmetric and transitive. Equality is the most familiar example of an equivalence relation. Other examples are 'has the same hair color as' and 'is the same age as'. The use of equivalence relations on a domain serves primarily to structure a domain into subsets whose members are regarded as equivalent with respect to that relation.

For every equivalence relation there is a natural way to divide the set on which it is defined into mutually exclusive (disjoint) subsets which are called *equivalence classes*. We write $[\![x]\!]$ for the set of all y such that $\langle x, y\rangle \in R$.

Thus, when R is an equivalence relation, $[\![x]\!]$ is the equivalence class which contains x. The relation 'is the same age as' divides the set of people into age groups, i.e., sets of people of the same age. Every pair of distinct equivalence classes is disjoint, because each person, having only one age, belongs to exactly one equivalence class. This is so even when somebody is 120 years old, and is the only person of that age, consequently occupying an equivalence class all by himself. By dividing a set into mutually exclusive and collectively exhaustive nonempty subsets we effect what is called a *partitioning* of that set.

Given a non-empty set A, a *partition* of A is a collection of non-empty subsets of A such that (1) for any two distinct subsets X and Y, $X \cap Y = \emptyset$ and (2) the union of all the subsets in the collection equals A. The notion of a partition is not defined for an empty set. The subsets that are members of a partition are called *cells* of that partition.

For example, let $A = \{a,b,c,d,e\}$. Then, $P = \{\{a,c\},\{b,e\},\{d\}\}$ is a partition of A because every pair of cells is disjoint: $\{a,c\} \cap \{b,e\} = \emptyset$, $\{b,e\} \cap \{d\} = \emptyset$, and $\{a,c\} \cap \{d\} = \emptyset$; and the union of all the cells equals A: $\bigcup\{\{a,c\},\{b,e\},\{d\}\} = A$.

The following three sets are also partitions of A:

(3–15) $P_1 = \{\{a,c,d\},\{b,e\}\}$
$P_2 = \{\{a\},\{b\},\{c\},\{d\},\{e\}\}$
$P_3 = \{\{a,b,c,d,e\}\}$

P_3 is the trivial partition of A into only one set. Note however that the definition of a partition is satisfied.

The following two sets are not partitions of A:

(3–16) $C = \{\{a,b,c\},\{b,d\},\{e\}\}$
$D = \{\{a\},\{b,e\},\{c\}\}$

C fails the definition because $\{a,b,c\} \cap \{b,d\} \neq \emptyset$ and D because $\bigcup\{\{a\},\{b,e\},\{c\}\} \neq A$

There is a close correspondence between partitions and equivalence relations. Given a partition of set A, the relation $R = \{\langle x,y\rangle \mid x$ and y are in the same cell of the partition$\}$ is an equivalence relation. Conversely, given a reflexive, symmetric, and transitive relation R in A, there exists a partition of A in which x and y are in the same cell if and only if x and y are related by

R. The equivalence classes specified by R are just the cells of the partition. An equivalence relation in A is sometimes said to *induce a partition of* A.

As an example, consider the set $A = \{1,2,3,4,5\}$ and the equivalence relation

(3–17) $R = \{\langle 1,1\rangle,\langle 1,3\rangle,\langle 3,1\rangle,\langle 3,3\rangle,\langle 2,2\rangle,\langle 2,4\rangle,\langle 4,2\rangle,\langle 4,5\rangle,$
$\langle 4,4\rangle,\langle 5,2\rangle,\langle 5,4\rangle,\langle 5,5\rangle,\langle 2,5\rangle\}$

which the reader can verify to be reflexive, symmetric, and transitive. In this relation 1 and 3 are related among themselves in all possible ways, as are 2, 4, and 5, but no members of the first group are related to any member of the second group. Therefore, R defines the equivalence classes $\{1,3\}$ and $\{2,4,5\}$, and the corresponding partition induced on A is

(3–18) $P_R = \{\{1,3\},\{2,4,5\}\}$

Given a partition such as

(3–19) $Q = \{\{1,2\},\{3,5\},\{4\}\}$

the relation R_Q consisting of all ordered pairs $\langle x,y\rangle$ such that x and y are in the same cell of the partition is as follows:

(3–20) $R_Q = \{\langle 1,1\rangle,\langle 1,2\rangle,\langle 2,1\rangle,\langle 2,2\rangle,\langle 3,3\rangle,\langle 3,5\rangle,\langle 5,3\rangle,\langle 5,5\rangle,\langle 4,4\rangle\}$

R_Q is seen to be reflexive, symmetric, and transitive, and it is thus an equivalence relation.

Another example is the equivalence relation 'is on the same continent as' on the set $A = \{$France, Chile, Nigeria, Ecuador, Luxembourg, Zambia, Ghana, San Marino, Uruguay, Kenya, Hungary$\}$. It partitions A into three equivalence classes: (1) $A_1 = \{$France, Luxembourg, San Marino, Hungary$\}$, (2) $A_2 = \{$Chile, Ecuador, Uruguay$\}$ and (3) $A_3 = \{$Nigeria, Zambia, Ghana, Kenya$\}$.

3.5 Orderings

An *order* is a binary relation which is transitive and in addition either (i) reflexive and antisymmetric or else (ii) irreflexive and asymmetric. The former are *weak* orders; the latter are *strict* (or *strong*).

To illustrate, let $A = \{a, b, c, d\}$. The following are all weak orders in A:

(3–21) $R_1 = \{\langle a,b\rangle, \langle a,c\rangle, \langle a,d\rangle, \langle b,c\rangle, \langle a,a\rangle, \langle b,b\rangle, \langle c,c\rangle, \langle d,d\rangle\}$
$R_2 = \{\langle b,a\rangle, \langle b,b\rangle, \langle a,a\rangle, \langle c,c\rangle, \langle d,d\rangle, \langle c,b\rangle, \langle c,a\rangle\}$
$R_3 = \{\langle d,c\rangle, \langle d,b\rangle, \langle d,a\rangle, \langle c,b\rangle, \langle c,a\rangle, \langle a,a\rangle, \langle b,b\rangle, \langle c,c\rangle, \langle d,d\rangle, \langle b,a\rangle\}$

These are represented in Figure 3-3 as relational diagrams, from which it can be verified that each is indeed reflexive, antisymmetric, and transitive.

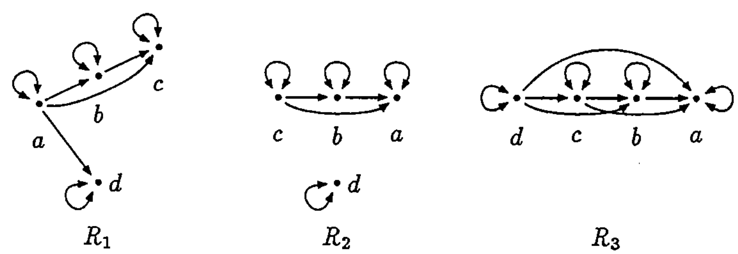

Figure 3–3:
Diagrams of the weak orders in (3-21)

To these weak orders there correspond the strict orders S_1, S_2 and S_3, respectively:

(3–22) $S_1 = \{\langle a,b\rangle, \langle a,c\rangle, \langle a,d\rangle, \langle b,c\rangle\}$
$S_2 = \{\langle b,a\rangle, \langle c,b\rangle, \langle c,a\rangle\}$
$S_3 = \{\langle d,c\rangle, \langle d,b\rangle, \langle d,a\rangle, \langle c,b\rangle, \langle c,a\rangle, \langle b,a\rangle\}$

These can be gotten from the weak orders by removing all the ordered pairs of the form $\langle x,x\rangle$. Conversely, one can make a strict order into a weak order by adding the pairs of the form $\langle x,x\rangle$ for every x in A.

As another example of an order, consider any collection of sets C and a relation R in C defined by $R = \{\langle X,Y\rangle \mid X \subseteq Y\}$ We have already noted in effect (Chapter 1, section 4) that the subset relation is transitive and reflexive. It is also antisymmetric, since for any sets X and Y, if $X \subseteq Y$ and $Y \subseteq X$, then $X = Y$ (this will be proved in Chapter 7). The corresponding strict order is the 'proper subset of' relation in C.

Further, we saw in Example (3-13) that the relation R 'greater than' in the set of positive integers is irreflexive, asymmetric and transitive. It is

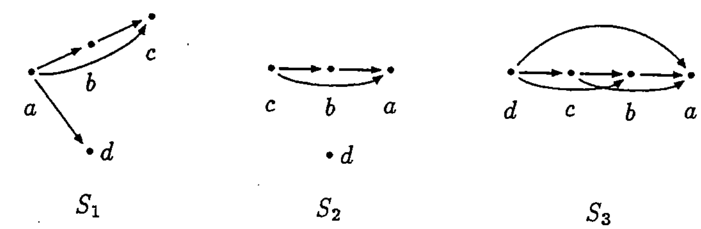

Figure 3–4:
Diagrams of the strict orders in (3-22)

therefore a strict order. (*Problem*: What relation defines the corresponding weak order?)

Some terminology: if R is an order, either weak or strict, and $\langle x,y\rangle \in R$, we say that x *precedes* y, x *is a predecessor of* y, y *succeeds* (or *follows*) x, or y *is a successor of* x, these being equivalent locutions. If x precedes y and $x \neq y$, then we say that x *immediately precedes* y or x is an *immediate predecessor* of y, etc., just in case there is no element z distinct from both x and y such that x precedes z and z precedes y. In other words, there is no other element between x and y in the order. Note that no element can be said to immediately precede itself since x and y in the definition must be distinct.

In R_1 and S_1 in (3-21) and (3-22), b is between a and c; therefore, although a precedes c, a is not an immediate predecessor of c. In R_2 and S_2, c is an immediate predecessor of b, and b is an immediate predecessor of a.

In diagramming orders it is usually simpler and more perspicuous to connect pairs of elements by arrows only if one is an immediate predecessor of the other. The remaining connection can be inferred from the fact that the relation is transitive. In order to distinguish weak from strict orders, however, it is necessary to include the 'reflexive' loops in weak orders. Diagrammed in this way, the orders in (3-21) would appear as in Figure 3-5. The diagrams of the corresponding strict orders would be identical except for the absence of the loops on each element.

There is also a useful set of terms for elements which stand at the extremes of an order. Given an order R in a set A,

1. an element x in A is *minimal* if and only if there is no other element

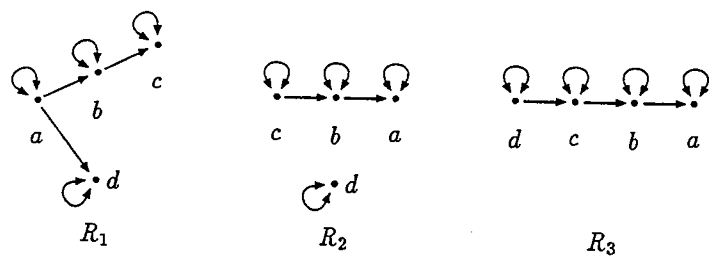

Figure 3–5: Immediate predecessor diagrams of the orders in (3-21)

in A which precedes x (examples: a in R_1 and S_1; c and d in R_2 and S_2; d in R_3 and S_3)

2. an element x in A is *least* if and only if x precedes every other element in A (examples: a in R_1 and S_1; d in R_3 and S_3)
3. an element x in A is *maximal* if and only if there is no other element in A which follows x (examples: c and d in R_1 and S_1; a and d in R_2 and S_2; a in R_3 and S_3)
4. an element x in A is *greatest* if and only if x follows every other element in A (examples: a in R_3 and S_3).

Note that a in R_1 and S_1 is both a minimal and a least element, while c and d in these same orders are both maximal but not greatest (c does not follow d, for example). Element d in R_2 and S_2 is both minimal and maximal but neither greatest nor least. The order defined by R in Example (3-13) has 1 as a maximal and greatest element (it follows all other elements and has no successors) but there is no minimal or least element in the order. Observe here that the form 'greatest' as used technically about orders need not coincide with the notions 'greater than' or 'greatest' in the realm of numbers.

A least element, if there is one in an order, is unique (if there were two, each would have to precede the other, and this would violate either asymmetry or antisymmetry), and similarly for a greatest element. There may be more than one minimal element, however (e.g., c and d in R_2 and S_2 above), and more than one maximal element. An order might have none of these; the relation 'greater than' in the set of all positive and negative integers and zero $\{0, 1, -1, 2, -1, \ldots\}$ has no maximal, minimal, greatest or least elements.

If an order, strict or weak, is also connected, then it is said to be a *total* or *linear* order. Examples are R_3 and S_3 above and the relation R of Example (3-13). Their immediate predecessor diagrams show the elements arranged in a single chain. Order R_1 is not total since d and c are not related, for example. Often orders in general are called *partial orders* or *partially ordered sets*. The terminology is unfortunate, since it then happens that some partial orders are total, but it is well established nonetheless, and we will sometimes use it in the remainder of this book.

Finally, we mention some other frequently encountered notions pertaining to orders. A set A is said to be *well-ordered* by a relation R if R is a total order and, further, every subset of A has a least element in the ordering relation. The set of natural numbers, $\mathbf{N} = \{0, 1, 2, 3, \ldots\}$ is well-ordered by the 'is less than' relation (it is a total order, and every subset of N will have a least element when ordered by this relation). The set of integers $\mathbf{Z} = \{0, 1, -1, 2, -2, \ldots\}$, on the other hand, is not well-ordered by that relation, since the negative integers get smaller 'ad infinitum'. Note that every finite linearly ordered set must be well-ordered.

A relation R in A is *dense* if for every $\langle x, y\rangle \in R$, $x \neq y$, there exists a member $z \in A$, $x \neq z$ and $y \neq= z$, such that $\langle x, z\rangle \in R$ and $\langle z, y\rangle \in R$. Density is an important property of the real numbers which we can think of as all the points lying on a horizontal line of infinite extent. The relation 'is greater than' is not dense on the natural numbers, but it is dense on the real numbers.

Exercises

1. **(a)** Determine the properties of the following relations on the set of all people. In each case, make the strongest possible statement, e.g., call a relation irreflexive whenever possible rather than nonreflexive.

   (i) is a child of
   (ii) is a brother of
   (iii) is a descendant of
   (iv) is an uncle of (assuming that one may marry one's aunt or uncle)

(b) Which of your answers would be changed if these relations were defined in the set of all male human beings?

2. Investigate the properties of each of the following relations. If any one is an equivalence relation, indicate the partition it induces on the appropriate set. (If you do not know the concepts, try to find some reasonable assumptions, state them explicitly, and do the exercise based on those).

   (a) $M = \{\langle x, y\rangle \mid x$ and y are a minimal pair of utterances of English$\}$
   (b) $C = \{\langle x, y\rangle \mid x$ and y are phones of English in complementary distribution$\}$
   (c) $F = \{\langle x, y\rangle \mid x$ and y are phones of English in free variation$\}$
   (d) $A = \{\langle x, y\rangle \mid x$ and y are allophones of the same English phoneme$\}$
   (e) Q is the relation defined by 'X is a set having the same number of members as Y' in some appropriate collection of sets.

3. Let $A = \{1,2,3,4\}$.

   (a) Determine the properties of each of the following relations, its inverse, and its complement. If any of the relations happens to be an equivalence relation, show the partition that is induced on A.

$$\begin{aligned}
R_1 &= \{\langle 1,1\rangle, \langle 2,1\rangle, \langle 3,4\rangle, \langle 2,2\rangle, \langle 3,3\rangle, \langle 4,4\rangle, \langle 4,1\rangle\} \\
R_2 &= \{\langle 3,4\rangle, \langle 1,2\rangle, \langle 1,4\rangle, \langle 2,3\rangle, \langle 2,4\rangle, \langle 1,3\rangle\} \\
R_3 &= \{\langle 2,4\rangle, \langle 3,1\rangle, \langle 3,4\rangle, \langle 2,2\rangle, \langle 1,3\rangle, \langle 4,3\rangle, \langle 4,2\rangle\} \\
R_4 &= \{\langle 1,1\rangle, \langle 2,4\rangle, \langle 1,3\rangle, \langle 2,2\rangle, \langle 3,1\rangle, \langle 4,4\rangle, \langle 3,3\rangle, \langle 4,2\rangle\}
\end{aligned}$$

   (b) Give the equivalence relation that induces the following partition on A: $P = \{\{1\},\{2,3\},\{4\}\}$.
   (c) How many distinct partitions of A are possible?

4. What is wrong with the following reasoning that reflexivity is a consequence of symmetry and transitivity? (Birkhof & MacLane (1965)). If $\langle x, y\rangle \in R$, then $\langle y, x\rangle \in R$, since we assume R is symmetric. If both $\langle x, y\rangle$ and $\langle y, x\rangle$ are in R, then $\langle x, x\rangle$ must be in R by transitivity.

5. Let $A = \{1,2,3,5,6,10,15,30\}$ and let R be a relation in A defined as follows:

$$R = \{\langle x, y\rangle \mid x \text{ divides } y \text{ without remainder}\}$$

   (a) List the members of R, and show that it is a weak partial order but not a total order.
   (b) Construct an immediate predecessor diagram for this order and identify any maximal, minimal, greatest, and least elements.
   (c) Do the same for the set $\wp(B)$, where $B = \{a,b,c\}$, and the relation 'is a subset of'.