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Brief article

Giving the boot to the bootstrap: How not to learn the natural numbers

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Abstract

According to one theory about how children learn the concept of natural numbers, they first determine that "one", "two", and "three" denote the size of sets containing the relevant number of items. They then make the following inductive inference (the *Bootstrap*): The next number word in the counting series denotes the size of the sets you get by adding one more object to the sets denoted by the previous number word. For example, if "three" refers to the size of sets containing three items, then "four" (the next word after "three") must refer to the size of sets containing three plus one items. We argue, however, that the Bootstrap cannot pick out the natural number sequence from other nonequivalent sequences and thus cannot convey to children the concept of the natural numbers. This is not just a result of the usual difficulties with induction but is specific to the Bootstrap. In order to work properly, the Bootstrap must somehow restrict the concept of "next number" in a way that conforms to the structure of the natural numbers. But with these restrictions, the Bootstrap is unnecessary.

Keywords: Numbers; Mathematics; Arithmetic; Bootstrapping; Induction

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1. Two number learners

Imagine that Fran and Jan are 3-year-old twins, eager to embark on learning math. Where they have got so far is this: They have learned to recite the number words to "nine". And they also know a little about what some of these words mean. They know that "one" can be used to refer to a property that collections have when they contain one object; they know, for example, that "there is one dog in the yard" is true just in case the yard contains one dog. And although it has taken them awhile to work this out, they have also learned that "two" similarly refers to a property that collections have when they contain two objects. Ditto for "three". What they know then are the relationships in (1):

(1) a. "One" refers to a property of collections containing one object.
b. "Two" refers to a property of collections containing two objects.
c. "Three" refers to a property of collections containing three objects.

Armed with the information in (1), Fran and Jan can help themselves to two cookies if an adult says, "You may have two cookies", and they can point to a scene in a picture book containing three horses if an adult says, "Show me the picture of three horses". They may have arrived at (1) in any of a number of ways. Perhaps they have a direct impression of the properties of oneness, twoness, and threeness from perception – a somewhat controversial process called subitizing (Mandler & Shebo, 1982; but see Balakrishnan & Ashby, 1992) – and have learned that the first three number words refer to these properties. Or they may have access to an internal quantity of activation whose magnitude increases directly with the number of objects in an array, attaching the number words to the properties occasioning these magnitudes (Deheane, 1997; Gallistel & Gelman, 1992; Wynn, 1992). Or they may have learned that "one", "two", and "three" refer to the property that arrays have when their attentional system is trained on one, two, or three simultaneous objects (Carey, 2001; Spelke, 2000). For our purposes, it won't matter how they come to know the facts in (1).[1]

Fran and Jan are now in a position to make an important discovery. They know that the list of number words they have memorized has a fixed order from "one" to "nine". And on the basis of (1), they can work out the fact that "two" refers to the

[1] On our view, it is unlikely that numerals denote properties of collections, for reasons that we have described elsewhere (Rips, Bloomfield, & Asmuth, 2005). So we doubt that (1) is true. But this assumption is almost universal in the literature on number acquisition and since this point is independent of the one we want to make about bootstrapping, we'll grant (1) here for the sake of the argument. Of course, it is not crucial that Jan and Fran can only count to 9 rather than to 20 or so, like most 3-year-olds. The limit at 9 makes the exposition easier, but for whatever number n a child can count to at the stage described here, we can arrive at the same conclusions by using mod_{n+1} in place of mod_{10} in the following development. It also would not matter much to our story exactly what Jan and Fran happen to believe about the meaning of words like "five" and "six" at this stage. They may think that "five" and "six" are both roughly synonymous with "some" or "a lot" (e.g., Carey, 2004) or, alternatively, they may believe that "five" and "six" each refer to distinct cardinal properties without being sure just which ones (Gelman & Butterworth, 2005; Sarnecka & Gelman, 2004).

property that collections have when one more object is added to the collections that "one" refers to. Similarly, "three" refers to the property that collections have when one more object is added to the collections that "two" refers to. It therefore looks suspiciously as if the general idea in (2) might be true:

(2) If "*k*" is a number word that refers to the property of collections containing *n* objects, then the next number word in the counting sequence "next(*k*)" refers to the property of collections containing one more than *n* objects.

Once Jan and Fran make the inductive leap from (1) to (2), they have mastered the "count-to-cardinal" transition (Fuson, 1988). They can work out the meaning of other number words, extending the facts in (1) to "four", "five", and beyond. With the help of a few additional principles (Gelman & Gallistel, 1978), they can also determine the numerosity or cardinality of a collection on their own by reciting the count sequence as they tick off the objects. Assuming that the meanings of the count terms are properties of collections in the sense of (1) and (2), then it seems we should credit Jan and Fran with an understanding of the natural numbers (0, 1, 2, 3, 4, ... or 1, 2, 3, 4, ..., depending on one's definition).

Or should we? Suppose that after another few months we check to see how Fran and Jan are getting along with their mathematics. What we discover is that, whereas Fran has been taught the familiar counting system in English, Jan has been taught a quite different one by a diabolic parent. If we ask how many cookies there are on a plate containing (as we would say) nine cookies, Fran and Jan both say "nine". If there are ten cookies on the plate, though, Fran says "ten" but Jan says "none". For 11 cookies, Fran says "eleven" and Jan says "one". For 21 cookies, Fran says "twenty-one" and Jan says "one". In short, when there are *n* cookies, Fran gives the English count term that we would give, but Jan gives the term corresponding to $\mathrm{mod}_{10}(n)$. (Here, $\mathrm{mod}_{10}(n)$ is the remainder you get after dividing *n* by 10.) Fran's system may seem the more natural one, since it is the one we usually use to enumerate things like cookies. But Jan's system is also quite intuitive in its own way and corresponds to telling time on a standard clock face (see Fig. 1a, below).[2]

So should we continue to credit Jan and Fran with knowledge of the natural numbers? In Jan's case, this seems quite incredible, since the counting system she uses does not have the properties of these numbers. For example, the first natural number

[2] Children of Jan's age may have trouble understanding "zero" (Wellman & Miller, 1986), so it may seem odd that our story assigns her such a concept. But, first, children at this age probably have less trouble with "no" (as in *Fred has no cookies*) or "none", which is all that is required for the meanings we are dealing with (Hanlon, 1988). Remember that we are pretending that the meaning of a number term is a cardinality (see Footnote 1). Second, and more important, we are not pretending that Jan's case models that of actual children – only that it serves to show that (2) does not immediately confer the concept NATURAL NUMBER. (As far as we know, there are no naturally-occurring, general-purpose counting systems with a modular structure, though of course cyclical systems do exist for hours of the day, days of the week, months of the year, and so on. There are also cultures with a few number words but without words for the full natural-number structure; see Gordon, 2004, and Pica, Lemer, Izard, & Dehaene, 2004, for recent examples.)

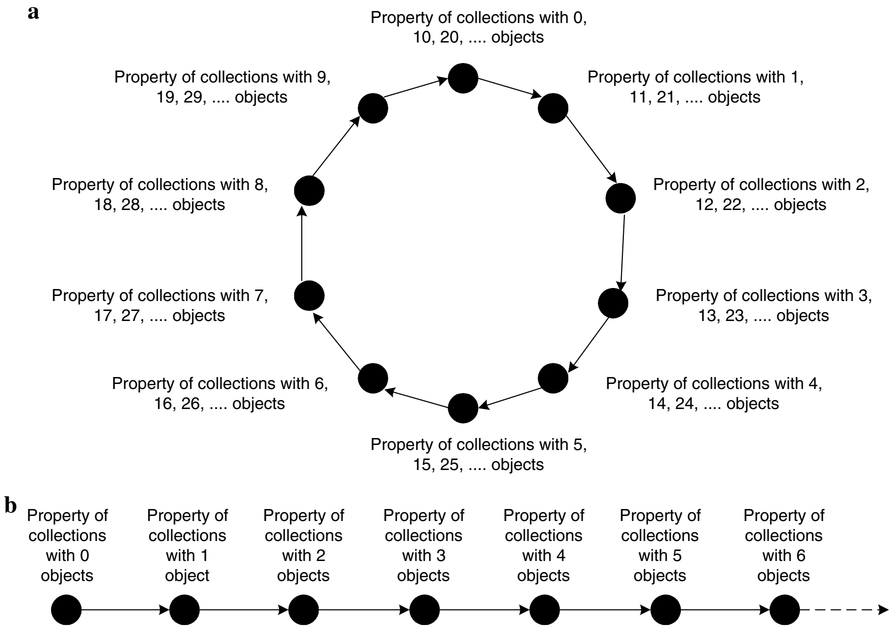

Fig. 1. (a) The internal structure of Jan's number system, and (b) the internal structure of Fran's number system.

(0 or 1, depending on the definition of natural numbers to which you subscribe) does not follow in sequence after any other number. Yet Jan's number term "none" (or "zero") follows directly after "nine". In fact, every term in her system follows after some other. Likewise, there are infinitely many distinct natural numbers. Yet Jan's system recognizes just ten. Although Jan's system is perfectly consistent and would even support (modular) arithmetic in Jan's future mathematical career, her system does not have the structure of the natural numbers. Has Jan reneged on the insight that she achieved in (2)? No. Jan follows the generalization in (2) just as rigorously as Fran. For any *n*,when there is a pile of *n* cookies, Jan says "there are *k*" cookies, where "*k*" corresponds to $\mathrm{mod}_{10}(n)$. When the pile is increased by one, she invariably says "there are next(*k*)" cookies, where "next(*k*)" is the next term in her counting sequence (i.e., "next(*k*)" corresponds to $\mathrm{mod}_{10}(n+1)$). The conclusion seems to be that the insight the girls achieved in inferring (2) from (1) does not by itself yield an understanding of the natural numbers.

If you still think that Jan has acquired the natural number concept through (2), that is probably because you're focusing on the fact that both Jan and Fran have the concept PROPERTIES OF COLLECTIONS THAT DIFFER BY ONE OBJECT. If these properties are the natural numbers, then Jan and Fran have concepts for them but use different numerals to refer to these numbers. According to this view, things are no different for the twins than if they happened to have been brought up speaking different natural languages (e.g., French and English) that have isomorphic numerals. Jan and Fran have notational variants for the same concepts, according to this theory.

This view, however, overlooks the difference in the organization of the numbers that Jan and Fran learn through their numerals. What Jan eventually learns as the result of her instruction is the following meaning rules:

(3) a. ‘‘none’’ refers to the property of collections containing 0 or 10 or 20 or ... objects.
b. ‘‘one’’ refers to the property of collections containing 1 or 11 or 21 or ... objects.
c. ‘‘two’’ refers to the property of collections containing 2 or 12 or 22 or ... objects.

And so on. The new rules for ‘‘one’’, ‘‘two’’, and ‘‘three’’ expand those in (1). Fig. 1a shows the complete set of Jan’s numbers in their cyclical order and illustrates the fact that in Jan’s system the numbers are also arranged in terms of PROPERTIES OF COLLECTIONS THAT DIFFER BY ONE OBJECT. For example, for any collection to which the property for ‘‘one’’ applies, adding one object produces a collection to which ‘‘two’’ applies. Likewise, for any collection to which the property for ‘‘nine’’ applies, adding one object produces a collection to which ‘‘none’’ applies, and so on. But the structure of these numbers clearly is not the structure of the natural numbers, which appears in Fig. 1b. There is no need to deny that Jan has the concepts PROPERTY OF COLLECTIONS CONTAINING ONE OBJECT, PROPERTY OF COLLECTIONS CONTAINING TWO OBJECTS, and so on. In Jan’s representation, however, these properties are not related in a way that yields the concept NATURAL NUMBER any more than they are related in a way that yields the concept MERSENNE PRIME, which some of these same properties also compose.[3]

2. Who believes in the bootstrap?

Let us call the inference from (1) to (2) the *Bootstrap* (Carey, 2004), since children making this inference are allegedly creating a concept of the natural number system where they had no such concept before. If you believe that children do not have an innate concept of the natural numbers, then the Bootstrap is an attractive proposition, since it appears to explain where these concepts come from. For example, Carey

[3] The description of Jan’s numbers in (3) and in Fig. 1a may look unnatural because of the disjunctions, but that is because we are describing them from our own point of view as natural-number chauvinists rather than as 1 mod_{10}, 2 mod_{10}, etc. Could a general constraint on word learning, such as mutual exclusivity (Markman, 1989), prohibit Jan from acquiring the structure in Fig. 1a? Mutual exclusivity might bias children away from the meaning rules in (3) by keeping them from using the same numeral for different sized collections. But it is unclear how mutual exclusivity operates in the case of abstract terms like numerals. Such a principle can’t be so stringent that it rules out applying ‘‘one’’ to one cookie, one person, and one mountain. Similarly, any general principle of word learning has to be compatible with learning cyclical terms for days of the week and months and seasons of the year. If children can learn days of the week, why not the system in Fig. 1a?

(2004, p. 67) proposes that children discover the concepts of the positive integers by interrelating the order of the number terms in the count sequence with the order of set sizes:

> Children may here make a wild analogy – that between the order of a particular quantity within an ordered list, and that between this quantity's order in a series of sets related by additional individuals. These are two quite different bases of ordering – but if the child recognizes this analogy, she is in the position to make the crucial induction. . .: If number word X refers to a set with cardinal value *n*, the next number word in the list refers to a set with cardinal value $n + 1$.

A second example occurs in James Hurford's (1987, pp. 125–127) ''Steps in the induction of a basic numerical lexicon and the concomitant number concepts''. The last two of these steps are:

(k) There is a parallel between the counting sequence (which the child now knows) and the elementary number rules just induced: *one* is followed by *two* in the counting sequence; and placing an object with an object (a oneness) results in a two-collection. Perhaps also: *two* is followed by *three* in the count sequence; and placing an object with a two collection results in a three-collection.

(l) Inductive generalization If X is followed by Y in the counting sequence, placing an object in an X-collection results in what is called a 'Y-collection.' Thus, what results from placing an object into a three-collection is called a 'four-collection' (new concept). And so on, as far as the conventional sequence of words stretches.

The passages we have just quoted are clearly and carefully formulated, and for this reason, it is obvious that they are variations on the Bootstrap in (2), but the same procedure probably lurks in many other theories of number concepts.[4]

[4] For example, Schaeffer, Eggleston, and Scott (1974, p. 377) make the following conjecture: ''Children may not have enough experience with arrays of 8, 9, and 10 objects to learn directly that 9 is greater than 8, and 10 greater than 9. Rather children may rely on the fact that 9 comes after 8 to judge the relative numerosity of the two cardinal numbers. That is, they integrate their knowledge of the relative numerosity of the cardinal numbers with which they have had direct experience, such as 5 and 6, with the knowledge of these numbers' position in the number series and generalize the knowledge to larger numbers''. Similarly, Bloom claims that ''In the course of development, children 'bootstrap' a generative understanding of number out of the productive syntactic and morphological structures available in the counting system. . . This knowledge emerges because (1) children first learn that the first words in the counting sequence map onto numerosities; (2) they then learn the specifics of the linguistic counting system; and (3) they map their nonlinguistic understanding of numerosities onto the linguistic structure of the number system. After the mapping takes place, children can *deduce* that the number system has the property of discrete infinity by noting that there is a one-to-one correspondence between numbers and number words and coming to realize that the counting system (in languages such as English) allows for the production of an infinity of strings. . .'' (Bloom, 1994, pp. 186–187, emphasis in original). Of course, in questioning the inference from (1) to (2), we don't mean to take a stand on the usefulness of bootstrapping procedures in acquiring the meaning of terms in other domains, such as verbs (see Fisher & Gleitman, 2002).

Of course, not everyone believes in the Bootstrap. For example, if you think that babies innately possess a system that embodies the properties of the natural numbers, then the Bootstrap is unnecessary. There may also be intermediate cases in which a mapping between systems modifies a pre-existing concept, conferring on it some additional property (e.g., discreteness) that it did not previously possess, thereby turning it into the natural number concept. Whether such proposals fall prey to the same difficulties as those discussed in Section 1 will depend on the nature of the structure-conferring and the structure-receiving systems.

3. Can the bootstrap be salvaged?

Fran and Jan's story makes it clear what is wrong with the Bootstrap as a theory of how children acquire concepts of the natural numbers. The problem is that "next(k)", the numeral in the counting sequence that comes immediately after "k" is not well defined. Unless you already know a counting sequence associated with the natural numbers, once you get past the numerals you have memorized, you do not automatically know how to continue. For Jan, "next(k)" is given by the mod_{10} system, but she might have been taught mod_{11}, mod_{12}, ..., or many other sequences that do not share the properties of the natural numbers. The Bootstrap will not produce the natural number concept in children because it incorrectly presupposes a system of numerals that tracks them. The generalization in (2) is correct but underdetermined.

At least, that is our diagnosis, but it is worth thinking about whether there is something in the vicinity of the Bootstrap that might overcome these difficulties. It is also worth considering how these difficulties differ from other puzzles about induction.

One response to the Bootstrapping problem, as we have stated it here, is that children do not in fact learn to count (i.e., recite the counting sequence "one", "two", ...) according to the mod_{10} system but they learn the standard sequence of numerals in their native language (provided it has one). The Fran–Jan problem is only a problem because someone threw Jan off track. In the usual, more benign, circumstances, no such problem would arise, and Jan would have learned the concept of the natural numbers via the Bootstrap. Jan herself would eventually come to see that something is wrong when she finds that her system runs into difficulties in communicating with others. So the Bootstrap is a perfectly fine heuristic for number learning.

However, this response misses the point of the example. The Bootstrap works for Fran and other kids in a count-friendly environment because when they learn how to extend the numeral sequence, they learn a sequence that is isomorphic to the natural numbers. In this sequence, "next(k)" in (2) exactly tracks the successor relation that *defines* the natural numbers. So it would not be the whole truth to say that Fran learns the natural number system through (2). If she uses (2) and ends up in the right place, it must be because she also manages to learn the structure of the counting sequence that corresponds to the natural numbers. This is what we might call *advanced counting* to distinguish it from the *simple counting* procedure of merely reciting the number terms to some fixed item, such as "nine" or "one hundred" (Rips et al., 2005). In advanced counting, you can always give the next numeral in the sequence from any starting point, whereas in simple counting you are stuck at the boundary number. The Bootstrap does not explain the development of advanced counting but simply presupposes it.[5]

A similar shoulder-shrugging response to the case of Fran and Jan assimilates it to well-known general problems about induction. Philosophers of science have recognized since Goodman (1955) that any body of data will typically support many inductive conclusions, conclusions that are equally consistent with the data but inconsistent with each other. For example, you can extrapolate the same set of data points in many different and mutually inconsistent ways. So additional (non-data) constraints are needed to explain both how people actually draw inductive conclusions and how they should draw them. Perhaps the difficulty with the Bootstrap is just another example of this type of indeterminacy. If so, then the Fran–Jan problem does not cast doubt on the Bootstrap per se but simply reflects the usual difficulty in justifying an inductive inference.

Of course, the inference from (1) to (2) *is* an inductive inference and so subject to the same uncertainties as others in this class. However, the problem we are focusing on is not a choice between rival inductive hypotheses. Fran and Jan are not debating whether they should spell out "next(k)" in terms of numerals for the natural numbers versus numerals for the mod_{10} numbers. This is because, at the point when they arrive at (2), neither one knows anything about either of these numeral sequences. These are not alternatives that are well-defined for them. There is, certainly, a question of how they will continue the sequence of number words following "nine", with Fran eventually going on to say "ten" and Jan to say "none". But this is still down the road at the point at which they "make a wild analogy" to (2), since they do not yet know either "ten" or "none". Neither word is part of their vocabulary at this stage. To put it another way, the problem with (2) is not that "next(k)" is ambiguous between rival numeral systems, it is that "next(k)" is completely vague outside the counting range "one" to "nine". So (2) cannot give them any guidance with natural number concepts.[6]

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[5] The development of advanced counting would take awhile if children had to rely solely on the English count system. Grinstead, McSwan, Curtiss, and Gelman (2005) claim that evidence for discrete infinity in this system does not emerge until after children have reached the term "one thousand". But perhaps information about advanced counting could come from written numerals or direct instruction. Grinstead et al. (2005) take the facts about the English count terms as support for their more general thesis that it is implausible that children could induce the recursive structure of the natural numbers from properties of natural language; instead, these investigators believe this information comes from an innate number module. Bootstrapping is therefore impossible because of the encapsulation of the language and number modules. For our purposes, however, we need not take a stand on modularity. Our own point is the more limited one that the natural number structure is not given by the correlation in (2).

[6] Much the same is true of another famous problem about extrapolation in arithmetic: Kripke's (1982) puzzle about the meaning of "plus". According to Kripke (channeling Wittgenstein), nothing about people's physical or mental make up determines whether by "plus" they mean the standard addition operator or some very different operator (called "quus") which is the same as ordinary addition for the problems they have computed so far but which uniformly equals 5 for larger problems. (E.g., if the largest problem computed to date is $435 + 981$, then $435 \oplus 981 = 1{,}416$, but $983 \oplus 992 = 5$, when "$\oplus$" is the quus function.) It is easy to see, however, that this problem, like Goodman's, is much broader than the one we have been discussing and applies equally to Jan and Fran.

4. Concluding comment

The Bootstrap will not give Jan and Fran the natural number concept, but it might not be irrelevant. Perhaps the Bootstrap is successful in convincing children that the integers – at least those within their counting range – are discrete, overcoming an initial dependence on a concept of continuous magnitude that appears to be common to infants and nonhuman animals (Deheane, 1997; Gallistel & Gelman, 1992; Wynn, 1992). And although we have stressed that the Bootstrap radically underdetermines the natural numbers, this leaves it open that a reinforced Bootstrap might work if we build in some restrictions. What might these be? Just these three items will suffice: First, there is a unique first term in the numeral sequence, say, "zero", which is never equal to "next(k)" (thereby eliminating the modular sequences). Second, if "next(k)" = "next(j)", then "k" = "j" (thereby eliminating other kinds of looping). And, finally, nothing else – nothing that cannot be reached from "zero" using "next" – can be part of the sequence. Once Jan and Fran have worked this out, they are ready to use the Bootstrap. But we would also be prepared to argue (Rips et al., 2005) that once they have worked this out they already have the natural number concept. The three constraints just mentioned are the axioms that define the natural numbers (Dedekind, 1888/1963). When Jan and Fran have learned them, they have no need of the Bootstrap.

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