Introduction to Semantics: Homework 3
Answer key

The semantics of conditional sentences

In our formulation of the semantics for sentential connectives, we gave functions from sets and pairs of sets to other sets as the meanings for *and*, *or*, and *not*.

Some notation:
• W is the set of worlds.
• $\mathscr{W}$ is the powerset of W, that is, the set of all subsets of W.
• A $\times$ B is the set of pairs of elements $\langle$a, b$\rangle$ from A and B respectively.
• 'f: A $\rightarrow$ B, such that C' means the function f whose domain is A and whose range is B, and of which C is true.

$[\![and]\!] = f\colon \mathscr{W} \times \mathscr{W} \rightarrow \mathscr{W}$, such that $f(\langle P, Q\rangle) = P \cap Q$

$[\![or]\!] = f\colon \mathscr{W} \times \mathscr{W} \rightarrow \mathscr{W}$, such that $f(\langle P, Q\rangle) = P \cup Q$

$[\![not]\!] = f\colon \mathscr{W} \rightarrow \mathscr{W}$, such that $f(P) = W - P$

These meanings give roles to *and*, *or*, *not* which are very similar to the roles of '$\wedge$', '$\vee$', and '$\neg$' in Classical logic. On this equivalence, '$\wedge$' maps to set intersection, '$\vee$' maps to set union, and '$\neg$' maps to set complementation.

Version 1

In an earlier class, we said that a sentence S1 entails a sentence S2 just in case the set of situations in which S1 is true if a subset of the set of situations in which S2 is true. Thinking of situations as worlds, we can say S1 entails S2 just in case $[\![\text{S1}]\!] \subseteq [\![\text{S2}]\!]$. Now consider the sentences from the assignment.

(1) a. If it is sunny, then it is warm.
    b. If Jane ate oatmeal, then she is happy.
    c. If Marco goes to the party, Maria will stay home.

Intuitively, we want the meanings of these sentences to be propositions in which the first clause entails the second clause. For example, for (1a), we want a proposition in which *It is sunny.* entails *It is warm.*. Perhaps, then, just like the connectives are defined in terms of intersection, union, and complementation, *if... then* should be defined in terms of the set-theoretic notion of subset. Unfortunately, subset does not correspond to a connective in Classical logic the way the other set-theoretic operations do; that is, 'A $\subseteq$ B' gives back a truth value, not another set. However, the following meaning which makes use of the notion of subset *does* give back another set of worlds.

(2) $[\![$*if... then*$]\!] = f\colon \mathscr{W} \times \mathscr{W} \rightarrow \mathscr{W}$, such that $f(\langle P, Q\rangle)$ is the largest subset S of W, such that $S \cap P \subseteq S \cap Q$

Now, we can say (2a) denotes the largest set of worlds S such that $[\![$*it is sunny*$]\!] \cap S \subseteq [\![$*it is warm*$]\!] \cap S$. Which worlds will be in this set? Let's say in some world w_1, it is sunny, but not warm. Then w_1 will not be in S since, if it were, then it would be in $[\![$*it is sunny*$]\!] \cap S$, but not $[\![$*it is warm*$]\!] \cap S$, failing the condition on S. It is possible to see that any world like w_1 will not be in S, and that all other worlds will be.

Version 2

Another way of treating conditional sentences would be to make direct use of the implicational connective '$\rightarrow$' from Classical logic by giving it a set-theoretic interpretation, as was done for the other connectives. '$\rightarrow$' can be defined in terms of '$\wedge$', '$\vee$', and '$\neg$' as $(p \rightarrow q) := ((\neg p) \vee q)$, and since we already have interpretations for the latter connectives, we can write the meaning of *if... then* as in (3).

(3) $[\![$*if... then*$]\!] = f\colon \mathscr{W} \times \mathscr{W} \rightarrow \mathscr{W}$, such that $f(\langle P, Q\rangle) = [\![$*or*$]\!](\langle [\![$*not*$]\!](P), Q\rangle)$

Which worlds will be in $[\![$*if... then*$]\!](\langle [\![$*it is sunny*$]\!], [\![$*it is warm*$]\!]\rangle)$? Just the ones that either are not in $[\![$*it is sunny*$]\!]$ or are in $[\![$*it is warm*$]\!]$ (or both). In fact, (2) and (3) are just different ways of specifying the same function.