At least some Determiners aren't Determiners

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1. The Classical Analysis...

One of the success stories in formal semantics is the analysis of NPs as generalized quantifiers, which can be traced back from Barwise & Cooper (1981) via Montague to Frege. The great attraction of this analysis is that it allows for a compositional analysis of the meaning of noun phrases that is consistent with independently motivated assumptions about their syntax. For example, Barwise and Cooper analyze NPs like *every boy*, *a boy*, *three boys* and *no boy* as having the syntactic structure [Det N], and interpretations as second-order predicates, as follows:

(1) a. $[\![every\ boy]\!] = \lambda P\lambda Q[P \subseteq Q]([\![boy]\!]), = \lambda Q[[\![boy]\!] \subseteq Q]$

b. $[\![a\ boy]\!] = \lambda P\lambda Q[P \cap Q \neq \emptyset]([\![boy]\!]), = \lambda Q[[\![boy]\!] \cap Q \neq \emptyset]$

c. $[\![three\ boys]\!] = \lambda P\lambda Q[\#(P \cap Q) \geq 3]([\![boy]\!]), = \lambda Q[\#([\![boy]\!] \cap Q) \geq 3]$

d. $[\![no\ boy]\!] = \lambda P\lambda Q[P \cap Q = \emptyset]([\![boy]\!]), = \lambda Q[[\![boy]\!] \cap Q = \emptyset]$

These meanings can be combined with the meanings of verbal predicates, like *left*, and we end up with representations like the following:

(2) a. $[\![every\ boy\ left]\!] = \lambda Q[[\![boy]\!] \subseteq Q]([\![left]\!]), = [\![boy]\!] \subseteq [\![left]\!]$

b. $[\![a\ boy\ left]\!] = \lambda Q[[\![boy]\!] \cap Q \neq \emptyset]([\![left]\!]), = [\![boy]\!] \cap [\![left]\!] \neq \emptyset$

c. $[\![three\ boys\ left]\!] = \lambda Q[\#([\![boy]\!] \cap Q) \geq 3]([\![left]\!]), = \#([\![boy]\!] \cap [\![left]\!]) \geq 3$

d. $[\![no\ boy\ left]\!] = \lambda Q[[\![boy]\!] \cap Q = \emptyset]([\![left]\!]), = [\![boy]\!] \cap [\![left]\!] = \emptyset$

Determiners are analyzed here as relations between a nominal predicate meaning and a verbal predicate meaning (e.g., $[\![every]\!] = \lambda P\lambda Q[P \subseteq Q]$).

This representation format has allowed researchers like Barwise and Cooper, van Benthem, Keenan and others to arrive at interesting and deep characterizations of the semantic nature of the determiners in natural language, such as conservativity, and of important subclasses, such as intersective and proportional determiners (cf. overviews like Westerståhl (1989) and Keenan & Westerståhl (1997)).

However, in spite of these impressive successes, the evidence is growing that the determiners of natural languages are not, or at least not all, relations between predicate meanings. Jacobs (1980) has argued that negative determiners, like German *kein* 'no', should be analyzed as indefinites with a negation that may have wider scope. In the analyses offered by the various versions of dynamic interpretation starting with Kamp (1981) and Heim (1982), a distinction is made between indefinite NPs like *a boy* and truly quantificational NPs like *every boy*; only the second type was analyzed as a quantifier. In the theories that tried to account for the singular/plural distinction and the collective and cumulative readings, such as Verkuyl (1981) and Link (1987), it was argued that number words like *three* should not be analyzed as determiners, but rather as adjectives. The GQ analysis also turned out problematic for certain types of NPs, such as predicative NPs. Partee (1987) proposed a system of type-shifts that would relate the various emerging interpretations of NPs to each other.

In this article I will discuss another class of NPs for which there is evidence that the GQ analysis is on the wrong track. These are NPs like *more than three boys*, *at most three boys*, *exactly three boys* and *between three and five boys*. The GQ analysis ascribes to the determiners of those NPs the following interpretations:

(3) a. $[\![more\ than\ three]\!] = \lambda P\lambda Q[\#(P \cap Q) > 3]$

b. $[\![$*at least three*$]\!] = \lambda P\lambda Q[\#(P \cap Q) \geq 3]$

c. $[\![$*at most three*$]\!] = \lambda P\lambda Q[\#(P \cap Q) \leq 3]$

d. $[\![$*exactly three*$]\!] = \lambda P\lambda Q[\#(P \cap Q) = 3]$

e. $[\![$*between three and five*$]\!] = \lambda P\lambda Q[3 \leq \#(P \cap Q) \leq 5]$

Obviously, this gives us the right truth conditions for sentences like *at most three boys left*; they say that the cardinality of the set that contains the boys that left is smaller or equal than three:

(4) $[\![$*at most three boys left*$]\!] = \#([\![$*left*$]\!] \cap [\![$*boys*$]\!]) \leq 3]$

However, once we consider a wider range of phenomena, problems appear with this type of representation.

2. ... and its Problems

2.1. *Three* and *at least three*

One problem of the classical analysis is that it cannot explain the semantic difference between a simple number determiner, like *three*, and the more complex determiner *at least three*. Barwise & Cooper (1981) analyze *three* as in (1.c), which allows for more than three elements in the intersection. This is justified by the fact that sentences like *Three boys left* are compatible with sentences that say that the number of boys that left was higher:

(5) Three boys left, perhaps even four.

The fact that a sentence like *three boys left* typically is understood as saying that no more than three boys left can be explained as a scalar implicature generated by the maxim of Quantity (Grice (1975)); it is actually a prime example for that type of implicature (cf. Horn (1972), Levinson (1983)). Speakers operate under the maxim of being as informative as possible (a maxim that is checked by other maxims, such as being concise). If the speaker had known that the number of boys that left was four, the speaker would have uttered the sentence *four boys left*; as the speaker did not do so, the hearer can safely assume that the speaker did not have the evidence for this latter sentence, or even knows that it is false.

But classical GQ theory assigns the same interpretation to *at least three boys*, cf. (3.b). This is semantically motivated, because the bare truth conditions seem to be the same in both cases: both *three boys left* and *at least three boys left* allow for the possibility that more than three boys left, and exclude the possibility that the number of boys that left was smaller than three. However, notice that *at least three boys* does not trigger the scalar implicature that we have observed with *three boys*. The sentence *at least three boys left* is not understood as implicating that no more than three boys left. Observe the following contrast:

(6) a. A: Three boys left.
B: No, four.

b. A: At least three boys left.
B: *No, four.

Also, we cannot really apply the usual locutions that cancel an implicature:

(7) ??At least three boys left, perhaps even four.

We might try to explain this rather striking difference by saying that expressions like *at least three boys* do not participate in Horn scales. It is well-known that scalar implicatures do not arise with just any terms. For example, we know that the term *apple* is a hyponym of *fruit*, and that whenever someone ate an apple, then he ate a fruit, but not vice versa. But a sentence like

(8) Peter ate a fruit.

certainly does not implicate that Peter didn't eat an apple, or even that the speaker doesn't know whether or not Peter ate an apple. Notice that we could apply usual quantity reasoning to this case: In uttering (8) Speaker has avoided the use of the (equally short and simple) expression *Peter ate an apple*, hence it follows that this expression is false. The answer to this problem is that not every two expressions α, β such that [...α...] has a stronger meaning than [...β...] are such that the utterance of [...β...] will generate the scalar implicature $\neg$[...α...]. For scalar implicatures to arise, α and β have to be, in addition, elements of a so-called Horn scale. Now, the number words *three* and *four* are elements of a Horn scale, but the nouns *fruit* and *apple* are not. This should presumably be part of the lexical knowledge of speaker and hearer.

But the suggestion that phrases like *at least three boys* do not participate in Horn scales is problematic. If number words form Horn scales, then they should do so in any context in which they appear. In particular, they should do so not only when they occur as determiners, as in *three boys*, but also when they occur as noun modifiers (or perhaps as parts of determiners), as in *at least three boys*. But then we predict that a sentence like *at least three boys left* has to be compared with semantically stronger sentences, like *at least four boys left*, and the fact that the speaker did not use these sentences leads to the implicature that they are false. We get exactly the same implicature as with *three boys left*.

Another option we have to block the scalar implicature with *at least three boys* is to assume that such phrases signal that the speaker does not know or does not want to say the precise number. As scalar implicature in its strong form, which negates the stronger alternatives of an assertion, requires that the speaker has complete knowledge of the subject matter, this would have the consequence that scalar implicature cannot arise in this case. But what could the status of this signal be? It could be a conventional implicature, or presupposition. However, it seems more plausible that the inference that the speaker does not know the precise number is a consequence of the fact that the speaker chose *at least three boys* over *three boys*, an expression that typically creates the implicature that the precise number is three, and hence that the speaker knows the precise number.

2.2. Cumulative Interpretations

The next problem concerns the representation of quantificational determiners with number words in general, be it *three*, *at least three*, *at most three* or *exactly three*. It is well known that they do not give us the right analysis for a prominent reading of sentences in which more than one such NP occurs, namely, the cumulative reading (cf. Scha (1981)):

(9) a. Three boys ate seven apples.

b. At least three boys ate at least seven apples.

c. At least three boys ate at most seven apples.

d. Exactly three boys ate exactly seven apples.

Under the cumulative interpretation, (9.a) means 'three boys ate apples and seven apples were eaten by boys'. The sentence does not tell us anything about how the apples distribute over the boys.[1] It is impossible to generate this reading using the GQ interpretation of the NPs in a principled way from the surface structure of this sentence, or from a logical form of the type that is normally assumed. The reason is that in the GQ interpretation, each NP has scope, and one NP must take scope over the other. We get interpretations like the following one, under the assumption that NPs

[1] In case the two numbers are the same, an interpretation that involves a one-to-one mapping is strongly preferred, as in *three boys ate three apples*. But I think this is not a separate reading of such sentences. Rather, it indicates a preferred cognitive model for how such sentences are understood. The reason of this preference certainly lies in the cognitive simplicity of one-to-one mappings.

undergo LF-movement (alternatively, we can give an in-situ interpretation that works with type shifting).

(10) *three boys*$_1$ [*seven apples*$_2$ [t$_1$ *ate* t$_2$]]
$\lambda P[\#([\![boy]\!] \cap P) \geq 3](\{x \mid [\lambda P[\#([\![apple]\!] \cap P) \geq 7](\{y \mid [[\![ate]\!](y)(x)]\})]\})$
$= \#([\![boy]\!] \cap \{x \mid \#([\![apple]\!] \cap \{y \mid [\![ate]\!](y)(x)\}) \geq 7\}) \geq 3\}$

This says that three boys ate seven apples each. The other possible derivation is one in which the object outscopes the subject, which amounts to saying that seven apples were eaten by three boys each. This reading is not only implausible for practical reasons, but also highly unlikely, arguably impossible, for linguistic reasons.

The paraphrase that I have used above rather suggests the following interpretation:

(11) $\#([\![boy]\!] \cap \{x \mid \exists y[[\![apple]\!](y) \wedge [\![ate]\!](x,y)]\}) \geq 3 \wedge$
$\#([\![apple]\!] \cap \{y \mid \exists x[[\![boy]\!](x) \wedge [\![ate]\!](x,y)]\}) \geq 7$

The problem is, as stated above, that it is unclear how the representation (11) can be developed from (9.a) in a principled, compositional way.

Incidentally, in Scha's original treatment of cumulative readings, the assigned representation is not quite the one in (11), but one in which the greater-or-equal signs are replaced by equal signs. However, strengthening "≥" to "=" should not be part of the truth conditions proper, but be derived as a scalar implicature. Notice that this additional meaning component can be canceled, which is characteristic for implicatures. This is particularly clear with the object NP (cf. (12.a)), a bit less so for the subject NP(cf. b), presumably because the subject NP is most likely interpreted as the topic.

(12) a. Three boys ate seven apples, perhaps even eight.

b. $^?$Three boys, perhaps even four boys, ate seven.

In any case, under Scha's original representation it remains unclear how (9.a) differs from (9.d), which presumably expresses Scha's representation.

We find cumulative readings with other NPs as well, for example, with *at least three boys* (cf. (9.b)). Examples like (9.c) which combines upward-entailing and downward-entailing quantifiers. Such cases occur in particular in sentences that describe statistical distributions:

(13) In Guatemala, (at most) three percent of the population owns (at least) seventy percent of the land.

Interestingly, the parts *at most* and *at least* are not even necessary here to get the intended interpretation. This clearly is a problem if we analyze, for example, *three percent* as *at least three percent*, as GQ theory wants to have it.

The problem cases discussed here clearly require a representation in which NPs are not scoped with respect to each other. Rather, they ask for an interpretation strategy in which all the NPs in a sentence are somehow interpreted in parallel, which is not compatible with our usual conception of the syntax/semantics interface which enforces a linear structure in which one NP takes scope over another.

2.3. Accent and Syntactic Distribution

A further problem of the GQ account of expressions like *at least* and *less than* is that their semantic contribution in sentences is influenced by accent in a way that other quantifiers, like *every*, *most* and *each*, are not. In the following example, the position of the main accent is indicated by acute accent on the stressed syllable.

(14) a. At least thrée boys left.

b. At least three bóys left.

(14.a) means: 'The number of boys that left is at least three'; it allows for there being a number n greater than three such that 'n boys left' is true. Clearly, accent identifies the number word here. The accent in (14.b) can either identify the head noun, or the construction consisting of number word and noun (a case of focus projection). Let us concentrate here on the second case. Then (14.b) says: 'The persons that left include three boys'; it allows for there being other persons x such that 'x left' is true as well. This difference leads to testable consequences when we look at the contexts in which (14.a) and (b) can be uttered. (14.a) but not (b) is fine as an answer to a question, *how many boys left?* And (14.b) but not (a) is fine as an answer to a question, *who (all) left?*

Such accent differences do not lead to comparable semantic differences with other quantifiers. Among the determiners that allow for number words are *all* and *the*. We do not find a similar difference in interpretation with the following examples:

(15) a. All thrée boys left.

b. All three bóys left.

Furthermore, expressions like *at least*, *at most*, *less than*, *more than* and *exactly* differ in their syntactic distribution from bona fide determiners like *every* or *most*. For example, they can form constituents with NPs, adjectives, VPs, or Det's, as the following examples show:

(16) a. John saw at least Mary.

b. The aggressors wanted more than the southern province.

(17) a. Mary was at least satisfied.

b. We are more than happy to serve you.

(18) a. The guest at least left early.

b. He at most spanked the child.

(19) a. At least some determiners aren't determiners.

b. At most eight percent of the students won't get a job.

Differences in syntactic distributions are normally cited to justify the exclusion of *only* from the set of natural-language quantifiers. This comes quite handy because an expression like *only boys*, analyzed as quantifiers, fails to show conservativity, which is typically seen as the most important general property of natural language determiners. But then expressions like *at least* or *more than* should not be analyzed as quantifiers either. Evidently, the usual GQ interpretation of expressions like *at least three boys* should be rather the result of the general meaning of *at least*, which can be combined with a wider variety of constituents, and *three boys*.

3. A New Analysis

In this section I will propose solutions to the various problems that we have encountered with the GQ analysis of expressions like *at least three boys*. I will start, however, with an analysis of NPs with simple number words like *three boys* in sentences with cumulative interpretations.

3.1. Number Words in Cumulative Interpretations

We have seen in section (2.2) that the GQ analysis of NPs like *three boys* is problematic for cumulative interpretations. In particular, this analysis forces us to assume that NPs are scoped, but the hallmark of cumulative interpretations is that the NPs of a sentence do not take scope over each other.

Cumulative interpretations need not be stipulated as a separate interpretation scheme as in Scha (1981); they can be derived in a more systematic and independently justified way. I have discussed this in Krifka (1989) and more specifically in Krifka (1992b); cf. also Landman (1995) and (to appear).

First, we have to assume a general rule for the interpretation of verbal predicates, which I have called **cumulativity**. With intransitive predicates α, cumulativity amounts to the following: If α applies to two individuals x and x′, then α also applies to the sum individual consisting of x and x′, for which I write $x \oplus x'$. For example, if the predicate *run* applies to Mary and to John, then it also applies to the sum consisting of Mary and John. This reflects the fact that the truth of the sentences *Mary runs* and *John runs* allows us to infer the truth of the sentence *Mary and John run*.

Transitive predicates behave the same. If the transitive predicate β applies to the individuals x and y, and also to the individuals x′ and y′, then it applies to the sum individuals $x \oplus x'$ and $y \oplus y'$ as well. For example, if Mary read 'Ulysses' and John read 'Moby-Dick', then the sentence

(20) Mary and John read 'Ulysses' and 'Moby-Dick'

is true as well. Of course, (20) has other readings as well, in particular the distributive reading in which both Mary and John are said to have read 'Ulysses' and 'Moby-Dick'. But this reading arguably requires an additional operator for distributivity that can be made explicit by *each*, as in *John and Mary each read 'Ulysses' and 'Moby-Dick'*. Furthermore, (20) in its cumulative reading may be interpreted more specifically as excluding that Mary read 'Moby-Dick' and John read 'Ulysses', an interpretation can be enforced by adding *respectively*. We may either account for this by an additional pragmatic principle (perhaps derived from Grices's manner maxim: 'be orderly'), or by assuming a sum formation that is order sensitive. Cumulativity based on order-sensitive sum formation would allow us to derive from $\beta(x, y)$ and $\beta(x', y')$ that $\beta(x \oplus x', y \oplus y')$, which is not equivalent to, say, $\beta(x \oplus x', y' \oplus y)$. However, for the rest of this paper I will disregard the issue how this more specific order-sensitive interpretation comes about, and I will assume that sum formation is commutative, that is, not order sensitive. The motivation for that is that we can continue (20) with *More specifically, Mary read 'Moby-Dick' and John read 'Ulysses'*, without contradiction.

Cumulativity can be generalized to n-place relations, in the following way:

(21) An n-place predicate R is cumulative iff the following holds:
If $R(x_1,...,x_n)$ and $R(x_1', ...,x_n')$, then $R(x_1 \oplus x_1',..., x_n \oplus x_n')$.

Cumulativity allows for inference patterns like the following. Assume that b_1, b_2, b_3 are boys and a_1 ... a_7 are apples.

(22) b_1 ate a_1 and a_2, b_2 ate a_3, a_4 and a_5, and b_3 ate a_6 and a_7,
then $b_1 \oplus b_2 \oplus b_3$ ate $a_1 \oplus a_2 \oplus a_3 \oplus a_4 \oplus a_5 \oplus a_6 \oplus a_7$.

It is natural to analyze an expression like *three boys* as a nominal predicate that applies to sum individuals that fall under the predicate *boys* and consist of three atoms. Then the second line of (22) entails the following:

(23) $\exists x[[\![\textit{three boys}]\!](x) \wedge \exists y[[\![\textit{seven apples}]\!](y) \wedge [\![\textit{eat}]\!](x,y)]]$

In general, we have that whenever the sentence schemes *n boys ate m apples* and *n′ boys ate m′ apples* are true, then the sentence schema *n+n′ boys ate m+m′ apples* is true. The reason for this is that *ate* is cumulative and that one and the same apple cannot be eaten twice. (Notice that this is a crucial condition that is not satisfied for, for example, the predicate *touch*).

I would now like to suggest that (23) is a representation of the sentence (9.a), repeated here.

(24) Three boys ate seven apples.

First of all, notice that (23), in contrast to (11), can be derived in a straightforward way from (24), We have to work under the assumption that an NP like *three boys* is analyzed as $[_{NP}\ \varnothing_{DET}\ [_{N}\ \textit{three boys}]]$, with an empty determiner position that is interpreted as an existential quantifier, and an analysis of *three* as a modifier that restricts the denotation of *boys* to those sum individuals that consist of three atoms, a representation suggested in Link (1987):

(25) a. LF: *three boys*$_1$ [*seven apples*$_1$ [t_1 *ate* t_2]]

b. $\lambda P \exists x[[\![\textit{three boys}]\!](x) \wedge P(x)](\lambda x[\lambda P \exists x[[\![\textit{seven apples}]\!](x) \wedge P(x)](\lambda y[[\![\textit{ate}]\!](x,y)]))$

= (23)

Alternatively, we can follow the analysis of indefinites in Heim (1982) and assume that the existential quantifier is introduced by global existential closure.

Notice that in the representation (23), the two NPs do not take scope with respect to each other, insofar as formulas of the form $\exists x[\Phi[x] \wedge \exists y[\Psi[x,y]]]$ are equivalent to formulas of the form $\exists x,y[\Phi[x] \wedge \Psi[x,y]]$, if y does not occur free in Φ. This is one important requirement for cumulative readings. Furthermore, notice that we get the same interpretation as with (11), under the assumption that $[\![\textit{eat}]\!]$ is cumulative. The formula (23) is not only true for the situation described in (22), but for any situation in which the number of boys that ate apples is at least three, and the number of apples eaten by boys is at least seven.

How does the additional information arise that we get from (24) and that is captured in Scha's original rendering of such sentences, namely, that the number of boys that ate apples is exactly three, and the number of apples eaten by boys is exactly seven? As suggested in section (2.2), we get this is by scalar implicature A speaker that utters (24) selects that sentence out of a competing set of other sentences, given schematically below:

(26)

$$\begin{Bmatrix} \ldots \\ \text{four} \\ \text{three} \\ \text{two} \\ \ldots \end{Bmatrix} \text{boys ate} \begin{Bmatrix} \ldots \\ \text{eight} \\ \text{seven} \\ \text{six} \\ \ldots \end{Bmatrix} \text{apples}$$

Grice's maxim of Quantity, more specifically its first submaxim, 'be as informative as required by the purpose of the information exchange', will force the speaker to choose the highest numbers *n*, *m* such that the sentence *n boys ate m apples* is true. This is because the sentence *n boys ate m apples* entails sentences like *n′ boys ate m′ apples*, for certain *n′* smaller than *n* and *m′* smaller than *m*, but is not entailed by them.[2] This is certainly not the case for all such alternative number words *n′*, *m′*. For example, the model in (22) does not support that two boys ate six apples. This is different from other known cases of scalar implicature, for which it is often assumed that, if a sentence $\Phi[\alpha]$ that contains α is true, were α is an element of a Horn scale, and β is any expression that is lower on the Horn scale, then $\Phi[\beta]$ will be true as well.[3] For example, the sentence *John ate seven apples* entails that *John ate six apples*, *John ate five apples*, and so on. However, if the main interest is in the number of boys that ate apples, and the number of apples eaten by boys, which is a very typical background for such sentences, then the maxim of Quantity will lead us to assume the greatest such numbers that still yield a true proposition.

That we need a special type of background for the understanding of such sentences becomes obvious when we consider sentences in cumulative interpretations that express statistical correlations, as in (13), repeated here.

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[2] One exception are sentences in collective intepretations (cf. Koenig 1991). Assume that Mary and Sue jointly own a horse, then the sentence *Two girls own a horse* is true, but the sentence *one/a girl owns a horse* isn't. This shows that the collective interpretation should be taken as a separate reading of a sentence that is not dependent on alternatives in the same sense as the cumulative interpretation.

[3] With the exception of hierarchical scales. For example, the sentence *Mary is a fúll professor* does not entail that Mary is an associate professor. I will come back to such scales.

(27) In Guatemala, three percent of the population own seventy percent of the land.

It is clear that the maximization strategy discussed for (24) does not work in this case. Under the simplifying (and wrong) assumption that foreigners do not own land in Guatemala, and all the land of Guatemala is owned by someone, this strategy would lead us to select the alternative *In Guatemala, 100 percent of the population own 100 percent of the land*, which clearly is not the most informative one among the alternatives — as a matter of fact, it is pretty uninformative. We cannot blame this on the fact that the NPs in (27) refer to percentages, as we could equally well express a similar statistical generalization with the following sentence (assume that Guatemala has 10 million inhabitants and has an area of 100,000 square kilometers):

(28) In Guatemala, 300,000 inhabitants own 70,000 square kilometers of land.

Again, the alternative *In Guatemala, 10 million inhabitants own 100,000 square kilometers of land* would be uninformative, under the background assumptions given.

What is peculiar with sentences like (27) is that they want to give information about the bias of a statistical distribution. One conventionalized way of expressing particularly biased distributions is to select a small set among one dimension that is related to a large set of the other dimension. Obviously, to characterize the distribution correctly, one should try to decrease the first set, and increase the second. In terms of informativity of propositions, if (27) is true, then there will be alternative true sentences of the form *In Guatemala, n percent of the population own m percent of the land*, where *n* is greater than *three*, and *m* is smaller than *seventy*. But these alternatives will not entail (27), and they will give a less accurate picture of the skewing of the land distribution.

3.2. The Generation and Use of Alternatives

One important point for our purposes is that scalar implicature cannot be applied locally in cases like (24), but has to be applied globally, across the board. It has to affect the two positions that introduce alternation simultaneously, as illustrated in (26). Which type of grammatical process would provide us with the necessary information to express this type of access to variation?

One way is to assume the mechanism that Rooth (1985) has suggested for the description of focus-sensitive operators, like *only* and *even*. In this theory, expressions that are in focus have the semantic function of introducing alternatives. These alternatives are then projected in a systematic way, till they meet an operator that makes use of them. As I said, Rooth considered alternatives introduced by focus, which are marked by accent. While we find that accent on the number words is quite natural in cases like (24) and (27), it does not seem to be required to get the indicated cumulative interpretations. So we might assume that certain types of expressions, like number words, can introduce alternatives without the help of focus. Also, their alternatives are of a particular type; the number adjective *three* does not introduce a color adjective like *green* as an alternative, but other number adjectives. This property of introducing alternatives without the help of focus is characteristic for the expressions that have been considered as being part of a Horn scale.[4]

Rooth (1985) proposed a general and simple mechanism under which alternatives are projected. Assume that the standard semantic interpretation for a complex expression $[\alpha\ \beta]$ is given compositionally as a function f of the meaning of the immediate parts α and β. Then the set of alternatives of the interpretation of $[\alpha\ \beta]$ is the set of meanings that can be obtained by applying the function f to the alternatives of the meaning of α and of β. Let us refer to the alternatives of the meaning of α with $[\![\alpha]\!]_A$, then we have the following general rule:

(29) If $[\![[\alpha\ \beta]]\!] = f([\![\alpha]\!], [\![\beta]\!])$, then $[\![[\alpha\ \beta]]\!]_A = \{f(X, Y) \mid X \in [\![\alpha]\!]_A, Y \in [\![\beta]\!]_A\}$

[4] In Krifka (1995) I argue that negative polarity items, like *ever* or *lift a finger*, have this property as well.

Expressions that do not have proper alternatives are assumed to have the singleton set containing their meaning as alternatives. The scheme in (29) allows for the simultaneous introduction of proper alternatives for α and β, which results in the type of variation illustrated in (26). To see this, consider the derivation of the alternatives of (24), along the lines of (25). Here, I render number words by numbers, where, e.g., 7(x) says that x is a sum individual consisting of seven atoms (it can be seen as an abbreviation of $\lambda x[\#(x) = 7]$, where #(x) gives the number of atoms that the sum individual x consists of). N is the set of all number words.

(30) a. $[\![\textit{seven}]\!] = \lambda P\lambda x[7(x) \wedge P(x)]$
$[\![\textit{seven}]\!]_A = \{\lambda P\lambda x[n(x) \wedge P(x)] \mid n \in N\}$

b. $[\![\textit{apples}]\!] = \text{APPLES}$
$[\![\textit{apples}]\!]_A = \{\text{APPLES}\}$

c. $[\![\textit{seven apples}]\!]$
$= \lambda P\lambda x[7(x) \wedge P(x)](\text{APPLES}),$
$= \lambda x[7(x) \wedge \text{APPLES}(x)]$
$[\![\textit{seven apples}]\!]_A$
$= \{X(Y) \mid X \in [\![\textit{seven}]\!]_A, Y \in [\![\textit{apples}]\!]_A\}$
$= \{\lambda x[n(x) \wedge \text{APPLE}(x)] \mid n \in N\}$

d. $[\![\varnothing_{DET}]\!] = \lambda Q\lambda P\exists x[Q(x) \wedge P(x)]$
$[\![\varnothing_{DET}]\!]_A = \{\lambda Q\lambda P\exists x[Q(x) \wedge P(x)]\}$

e. $[\![[_{NP} \varnothing\ [\textit{seven apples}]]]\!]$
$= \lambda Q\lambda P\exists x[Q(x) \wedge P(x)](\lambda x[7(x) \wedge \text{APPLES}(x)])$
$= \lambda P\exists x[7(x) \wedge \text{APPLES}(x) \wedge P(x)]$
$[\![[_{NP} \varnothing\ [\textit{seven apples}]]]\!]_A$
$= \{X(Y) \mid X \in [\![\varnothing_{DET}]\!]_A, Y \in [\![\textit{seven apples}]\!]_A\}$
$= \{\lambda P\exists x[n(x) \wedge \text{APPLE}(x) \wedge P(x)] \mid n \in N\}$

f. $[\![[t_1\ \textit{ate}\ t_2]]\!] = \text{ATE}(x_1, x_2)$
$[\![[t_1\ \textit{ate}\ t_2]]\!]_A = \{\text{ATE}(x_1, x_2)\}$

g. $[\![[[_{NP} \varnothing\ [\textit{seven apples}]]_2\ [t_1\ \textit{ate}\ t_2]]]\!]$
$= \lambda P\exists x[7(x) \wedge \text{APPLES}(x) \wedge P(x)](\lambda x_2[\text{ATE}(x_1, x_2)])$
$= \exists x[7(x) \wedge \text{APPLES}(x) \wedge \text{ATE}(x_1, x)]$
$[\![[[_{NP} \varnothing\ [\textit{seven apples}]]_2\ [t_1\ \textit{ate}\ t_2]]]\!]_A$
$= \{X(\lambda x_2[Y]) \mid X \in [\![[_{NP} \varnothing_{DET}\ [\textit{seven apples}]]]\!]_A, Y \in [\![[t_1\ \textit{ate}\ t_2]]\!]\}$
$= \{\exists x[n(x) \wedge \text{APPLES}(x) \wedge \text{ATE}(x_1, x)] \mid n \in N\}$

h. $[\![[_{NP} \varnothing\ [\textit{three boys}]]]\!] = \lambda P\exists x[3(x)\ \text{BOYS}(x) \wedge P(x)]$
$[\![[_{NP} \varnothing\ [\textit{three boys}]]]\!]_A = \{\lambda P\exists x[n(x)\ \text{BOYS}(x) \wedge P(x)] \mid n \in N\}$

k. $[\![[[_{NP} \varnothing\ [\textit{three boys}]]_1\ [[_{NP} \varnothing\ [\textit{seven apples}]]_2\ [t_1\ \textit{ate}\ t_2]]]]\!]$
$= \lambda P\exists x[3(x)\ \text{BOYS}(x) \wedge P(x)](\lambda x_1\exists x[7(x) \wedge \text{APPLES}(x) \wedge \text{ATE}(x_1, x)])$
$= \exists y\exists x[3(y) \wedge \text{BOYS}(y) \wedge 4(x) \wedge \text{APPLES}(x) \wedge \text{ATE}(y, x)]$
$[\![[[_{NP} \varnothing\ [\textit{three boys}]]_1\ [[_{NP} \varnothing\ [\textit{seven apples}]]_2\ [t_1\ \textit{ate}\ t_2]]]]\!]_A$
$= \{X(\lambda x_1[Y]) \mid X \in [\![[_{NP} \varnothing\ [\textit{three boys}]]]\!]_A,$
$Y \in [\![[[_{NP} \varnothing\ [\textit{seven apples}]]_2\ [t_1\ \textit{ate}\ t_2]]]\!]_A\}$
$= \{\exists y\exists x[n(y) \wedge \text{BOYS}(y) \wedge m(x) \wedge \text{APPLES}(x) \wedge \text{ATE}(y, x)] \mid n, m \in N\}$

We see that the variations introduced by *seven* and *three* end up on equal footing in the set of alternatives. This was intended in Rooth's original application of this idea, which included the treatment of sentences with complex foci, as in *John only introduced Bíll to Súe*, with the meaning 'the only pair $\langle x, y\rangle$ such that John introduced x to y is $\langle$Bill, Sue$\rangle$'.

The derivation (30) leads to the following pair of a regular meaning and a set of alternatives:

(31) a. Meaning: $\exists x\exists y[3(x) \wedge \text{BOYS}(x) \wedge 7(y) \wedge \text{APPLES}(y) \wedge \text{ATE}(x, y)]$

b. Alternatives: $\{\exists x \exists y[n(x) \wedge \text{BOYS}(x) \wedge m(y) \wedge \text{APPLES}(y) \wedge \text{ATE}(x, y)] \mid n, m \in N\}$

The meaning expresses the proposition 'three boys ate seven apples', and the set of alternatives consist of propositions of the form 'n boys ate m apples', where n, m are numbers. I should say that I simplified things a bit for the sake of exposition, as I did not indicate in the semantic representations the possible world parameter. Actually, every semantic representation given here should be a function from possible worlds. For example, the meaning is not quite the one given in (31.a), but rather $\lambda w \exists x \exists y[3(x) \wedge \text{BOYS}(w)(x) \wedge 7(y) \wedge \text{APPLES}(w)(y) \wedge \text{ATE}(w)(x, y)]$, a function from worlds w to truth if, in w, three boys ate seven apples.

The formation of the scalar implicature discussed in section (3.1) has to relate this meaning and this set of alternatives in some way. It is plausible to locate the origin of scalar implicature in a pragmatic operator ASSERT that expresses that a sentence is asserted.[5] The function of this operator can be expressed in the following way (cf. also Krifka (1992a)):

(32) ASSERT(M, A, c) (a sentence with meaning M and alternatives A in a context c is asserted):
— the speaker claims M (in c);
— for every alternative $M' \in A$, $M' \neq M$, the speaker explicitly does not claim M′ (in c).

There must be a pragmatic reason why the speaker introduces alternative propositions without actually claiming them, reasons that are obvious to the hearer. There are, essentially, two types of reasons for the cases that are relevant here:

- It may be that M′ is more informative than M (in c). The reason in this case is obviously that the speaker lacks evidence for M′, and perhaps even has evidence that M′ is false. That is, uttering M′ instead would violate the maxim of Quality.
- It may be that M′ is less informative than M (in c). The reason in this case is obviously that the speaker prefers M because it gives more information, in c. That is, uttering M′ instead would violate the maxim of Quantity.

It is important to have the utterance context, c, as a parameter in (32). This is because we have seen with the contrast of examples like (24) and (27) (the Guatemala example) that what counts as more or less informative might depend on the context.

3.3. *At least* Quantifiers

Let us now discuss expressions like *at least*, *at most* or *exactly* that we have identified as being subject to accentual influences in section (2.3). In this section I will outline the basics of the theory I am proposing with the example *at least*. In the next sections I will extend the treatment to other quantifiers, which will necessitate an important change in the theory that we will develop here.

We may try to analyze expressions like *at least three* in a way that is inspired by our treatment of simple number words in the preceding section. That is, we may analyze *at least three* as an adjective that applies to sum individuals which consist of at least three atoms, and *at most three* as an adjective that applies to sum individuals that consist of at most three atoms. But this analysis would not solve the problems discussed in section (2), and introduce problems of its own. First, it would predict, together with the assumptions we have made in section (3.2) that scalar implicature is invoked by a sentence like (33.a). It contains the number words *three* and *seven*, which by assumption introduce alternatives regardless in which syntactic position they occur.

(33) a. At least three boys ate at least seven apples.

b. Meaning: $\exists x,y[\geq 3(x) \wedge \text{BOYS}(x) \wedge \geq 7(y) \wedge \text{APPLES}(y) \wedge \text{EAT}(x,y)]$

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[5] Other illocutionary operators, like interrogative or imperative, will lead to similar scalar implicatures. The proposal that illocutionary operators may be sensitive to the alternatives of a sentence goes back to Jacobs (1984), where it is stated for alternatives that are introduced by focus.

c. Alternatives: $\{\exists x,y[\geq n(x) \wedge \text{BOYS}(x) \wedge \geq m(y) \wedge \text{APPLES}(y) \wedge \text{EAT}(x,y)] \mid n,m \in N\}$

I have used the notation "≥3" as an abbreviation for $\lambda x[\#(x) \geq 3]$. The problem with (33) is that, if we apply scalar implicature as specified for the operator ASSERT, we will end up with exactly the same meaning as for (24). By scalar implicature, any utterance of (33.a) will exclude propositions of the form *at least n boys ate at least m apples*, for *n* greater than *three* and *m* greater than *seven*, which amounts to saying that the number of boys that ate apples was three, and the number of apples eaten by boys was seven.

A further problem for the application of ideas developed in section (3.2) is that the particle *at least* need not be related to a set of alternatives that are ordered by semantic strength. Example:

(34) Mary is at least an assóciate professor (perhaps even a full professor).

In the first clause of (34), the alternatives ('Mary is an assistant professor', 'Mary is an associate professor', 'Mary is a full professor' etc.) are not ordered with respect to semantic strength. We must assume that particles like *at least* presuppose that the alternatives are ranked in some way which sometimes is related to semantic strength (as in *Mary ate at least thrée apples*), but often not, as in (34).

I will represent ranked alternatives here as partial ordering relation ≤, from which we can derive the set of alternatives as the field of ≤, defined as follows:

(35) $\text{Field}(\leq) = \{x \mid \exists y[x \leq y \vee y \leq x]\}$

Expressions like number words, terms denoting ranks in a social hierarchy, and many other expressions come with such an ordering relation. One important ordering relation is induced by the part relation of sum individuals, $\leq_i$, which tells us, for example, that John is a part of the sum individual consisting of John and Mary, and this is a part of the sum individual consisting of John, Mary and Sue. This ordering relation is responsible for the use of *at least* in sentences like the following:

(36) At least John and Mary left (perhaps also Sue, i.e., perhaps even John, Mary and Sue).

In the previous section, example (30), we have seen how alternatives are projected in semantic compositions. If the alternatives are ordered, the ordering should be projected in a similar way. This can be done with the following projection rule, which expands on (29) and says that the resulting ordering relation is a combination of the ordering relations of the parts.

(37) If $[\![[\alpha\ \beta]]\!] = f([\![\alpha]\!], [\![\beta]\!])$,
then $[\![[\alpha\ \beta]]\!]_A = \{\langle f(X, Y), f(X', Y')\rangle \mid \langle X, X'\rangle \in [\![\alpha]\!]_A \text{ and } \langle Y, Y'\rangle \in [\![\beta]\!]_A\}$

Let me discuss the theory proposed here with a simple example:

(38) At least thrée$_F$ boys léft.

Accent on *three* indicates focus on the number word, here indicated by a subscript F. Numbers are related to an order relation $\leq_N$.

(39) a. $[\![\textit{thrée}_F]\!] = \lambda P\lambda x[3(x) \wedge P(x)]$
$[\![\textit{thrée}_F]\!]_A = \{\langle \lambda P\lambda x[n(x) \wedge P(x)], \lambda P\lambda x[m(x) \wedge P(x)]\rangle \mid n \leq_N m\}$

Expressions that are not related to any ordering relation and that are not in focus do not introduce proper alternatives. We can assume that they come with a set consisting of their meaning proper (for unordered alternatives), and a pair formed with their meaning proper (for ordered alternatives). I will call such alternatives **standard alternatives**, for short.

(40) b. $[\![\textit{boys}]\!] = \text{BOYS}$
$[\![\textit{boys}]\!]_A = \{\text{BOYS}, \langle \text{BOYS}, \text{BOYS}\rangle\}$ (the standard alternatives).

c. $[\![\textit{thrée}_F\ \textit{boys}]\!] = \lambda x[3(x) \wedge \text{BOYS}(x)]$
$[\![\textit{thrée}_F\ \textit{boys}]\!]_A = \{\langle \lambda x[n(x) \wedge \text{BOYS}(x)], \lambda x[m(x) \wedge \text{BOYS}(x)]\rangle \mid n \leq_N m\}$

Let us assume that the particle *at least* can be applied at this level, on nominal predicates (notice that such phrases can occur in predicate position, as in *these are at least twenty eggs*). I suggest that the particle *at least*, when applied to α, constructs a new meaning as the union of the alternatives of α that stand in relation to the meaning of α. The alternatives then are just the singleton set, that is, no new alternatives are projected.

(41) $[\![\textit{at least } \alpha]\!] = \bigcup \{P \mid \langle [\![\alpha]\!], P\rangle \in [\![\alpha]\!]_A\}$
$[\![\textit{at least } \alpha]\!]_A$ = the standard alternatives

Here, $\bigcup$ S stands for the semantic union of the elements in the set S. This is the same as $\cup$ S, if the elements of S are given as sets. For example, we have

$$\cup\{\{x \mid \text{BOY}(x)\}, \{x \mid \text{GIRL}(x)\}\} =$$
$$\{x \mid \text{BOY}(x)\} \cup \{x \mid \text{GIRL}(x)\} =$$
$$= \{x \mid \text{BOY}(x) \vee \text{GIRL}(x)\}$$

The operation $\bigcup$ should have the same effect if the meanings are functions. For example, we would like to have

$$\bigcup\{\lambda x[\text{BOY}(x)], \lambda x[\text{GIRL}(x)]\} =$$
$$\lambda x[\text{BOY}(x)] \bigcup \lambda x[\text{GIRL}(x)] =$$
$$\lambda x[\text{BOY}(x) \vee \text{GIRL}(x)]$$

This can be achieved if we define $\bigcup$, as a connective for functions, as the generalized join operation (cf. Keenan & Faltz (1985)):

(42) a. If Φ, Ψ are sentences (type t), then $[\![\Phi]\!] \bigcup [\![\Psi]\!] = [\![\Phi \vee \Psi]\!]$

b. If α, β are expressions of type $\langle\sigma, \tau\rangle$, then $[\![\alpha]\!] \bigcup [\![\beta]\!] = \lambda X[[\![\alpha]\!](X) \bigcup [\![\beta]\!](X)]$

c. If S is a set of meanings of a type that can be conjoined by $\bigcup$,
then $\bigcup$ S is the result of conjoining all the elements of S by $\bigcup$.

For our example we get the following interpretation:

(39) b. $[\![\textit{at least } [\textit{thrée}_F \textit{ boys}]]\!]$
$= \bigcup \{P \mid \langle\lambda x[3(x) \wedge \text{BOYS}(x)], P\rangle \in$
$\{\langle\lambda x[n(x) \wedge \text{BOYS}(x)], \lambda x[m(x) \wedge \text{BOYS}(x)]\rangle \mid n \leq_N m\}\}$
$= \bigcup \{\lambda x[m(x) \wedge \text{BOYS}(x)] \mid 3 \leq_N m\}$
$= \lambda x[\geq 3(x) \wedge \text{BOYS}(x)]$
$[\![\textit{at least } [\textit{thrée}_F \textit{ boys}]]\!]_A = \{\langle\lambda x[\geq 3(x) \wedge \text{BOYS}(x)], \lambda x[\geq 3(x) \wedge \text{BOYS}(x)]\rangle\}$

And we arrive at the following result:

(39) c. $[\![[[\varnothing \textit{ at least } [\textit{three}_F \textit{ boys}]]_1 [t_1 \textit{ left}]]]\!] = \exists x[\geq 3(x) \wedge \text{BOYS}(x) \wedge \text{LEFT}(x)]$
$[\![[[\varnothing \textit{ at least } [\textit{three}_F \textit{ boys}]]_1 [t_1 \textit{ left}]]]\!]_A$ = the standard alternatives of the above

This differs from the sentence *three boys left*, as no proper alternatives are generated, hence no scalar implicatures arise. This explains the different behavior that we observed with sentences like (6) and (7).

It may be worthwhile to check how the sentence *three boys left* is derived in the new framework in which alternatives are ordered. The changes are of a technical nature. Here is a derivation:

(43) a. $[\![\textit{three boys}]\!] = \lambda x[3(x) \wedge \text{BOYS}(x)]$
$[\![\textit{three boys}]\!]_A = \{\langle\lambda x[n(x) \wedge \text{BOYS}(x)], \lambda x[n(x) \wedge \text{BOYS}(x)]\rangle \mid n \leq_N m\}$

b. $[\![[[\varnothing \textit{ three boys}]_1 [t_1 \textit{ left}]]]\!]$
$= \exists x[3(x) \wedge \text{BOYS}(x) \wedge \text{LEFT}(x)]$
$[\![[[\varnothing \textit{ three boys}]_1 [t_1 \textit{ left}]]]\!]_A$
$= \{\langle\exists x[n(x) \wedge \text{BOYS}(x) \wedge \text{LEFT}(x)], \exists x[m(x) \wedge \text{BOYS}(x) \wedge \text{LEFT}(x)]\rangle \mid n <_N m\}$

We have to slightly change the definition of the ASSERT operator given in (32), as the alternatives now are a relation, and the operator has to refer to the field of that relation. I also skip the context parameter c here.

(44) ASSERT(M, A):
— the speaker claims M;
— for every alternative $M' \in \text{Field}(A)$, $M' \neq M$, the speaker explicitly does not claim M′.

If we apply this ASSERT operator to our example we get the same interpretation as before. The speaker claims that three boys left, and explicitly does not claim propositions of the form 'n boys left', where $n \neq 3$. If $n < 3$, these alternatives are not claimed because they are less informative. If $n > 3$, they are not claimed because the speaker lacks the necessary evidence or knows that they are false.

The crucial difference between a sentence like *three boys left* and *at least three boys left* is that only the first sentence provides proper alternatives for the illocutionary operator. In the second sentence, the alternatives that are introduced by focus are used by the particle *at least* to construct the meaning proper of the sentence. (*At least* behaves just like other focus-sensitive operators in this respect.) No alternatives are projected beyond that, and hence no scalar implicatures arise.

The analysis of *at least* quantifiers gives us the right result for sentences in which more than one of these quantifiers occur in a sentence, leading to a cumulative interpretation. To see this, consider example (9.b), repeated here (with focus on the number word):

(45) a. At least thrée boys ate at least séven apples.

The application of the interpretation rules discussed above will lead to the following meaning.

(45) b. $\exists y \exists x[\geq 3(y) \wedge \text{BOYS}(y) \wedge \geq 7(x) \wedge \text{APPLES}(x) \wedge \text{EAT}(y, x)]$

In the examples considered so far the focus was on a number word, *three*. But we have seen that *at least* can focus on other expressions too. Let us first consider the following case, in which *at least* is applied to a conjunction of two names.

(46) a. At least [Jóhn and Máry]$_F$ left.

Focus on *John and Mary* signals the presence of alternatives, and the particle *at least* presupposes that the alternatives are ordered. The relevant ordering relation here is the part relation on individuals, $\leq_i$. We then get the following interpretation:

(46) b. $[\![[_{PN} \textit{Jóhn and Máry}]_F]\!] = \text{JOHN} \oplus \text{MARY}$
$[\![[_{PN} \textit{Jóhn and Máry}]_F]\!]_A = \{\langle x, y\rangle \mid x \leq_i y\}$

c. Type lift from e to $\langle\langle e, t\rangle, t\rangle$:
$[\![[_{NP} \textit{Jóhn and Máry}]_F]\!] = \lambda P[P(\text{JOHN} \oplus \text{MARY})]$
$[\![[_{NP} \textit{Jóhn and Máry}]_F]\!] = \{\langle \lambda P[P(x)], \lambda P[P(y)]\rangle \mid x \leq_i y\}$

d. $[\![\textit{at least} [\textit{Jóhn and Máry}]_F]\!]$
$= \bigcup \{P \mid \langle \lambda P[P(\text{JOHN} \oplus \text{MARY})], P\rangle \in \{\langle \lambda P[P(x)], \lambda P[P(y)]\rangle \mid x \leq_i y\}$
$= \bigcup \{\lambda P[P(y)] \mid \text{JOHN} \oplus \text{MARY} \leq_i y\}$
$= \lambda P \exists y[\text{JOHN} \oplus \text{MARY} \leq_i y \wedge P(y)]$
$[\![\textit{at least} [\textit{Jóhn and Máry}]_F]\!]_A$ = the standard alternatives

e. $[\![[[\textit{at least} [\textit{Jóhn and Máry}]_F]_1 [t_1 \textit{ left}]]]\!] = \exists y[\text{JOHN} \oplus \text{MARY} \leq_i y \wedge \text{LEFT}(y)]$
$[\![[[\textit{at least} [\textit{Jóhn and Máry}]_F]_1 [t_1 \textit{ left}]]]\!]_A$ = the standard alternatives

In step (c) we have performed a type shift from the type of entities to the type of quantifiers, which is necessary because the operation $\bigcup$ is not defined for type e. Notice that the alternatives and their ordering are projected in the usual fashion. We arrive at a meaning saying that a sum individual that contains John and Mary left.

Yet another case is presented by the following example:

(47) a. [Three bóys]$_F$ left.

b. At least [three bóys]$_F$ left.

What are the alternatives of *three boys*, and how are they ordered? It is natural to assume that the ordering is the one induced by the part relation on individuals, $\leq_i$, on sets, which I will call $\leq_I$.

(48) If P and Q are sets, then $P \leq_I Q$ iff $\forall x \in P \exists y \in Q[x \leq_i y]$

For example, we have 'three boys' $\leq_i$ 'four boys', and 'three boys' $\leq_i$ 'three boys and a girl'. With this ordering, (47.a) excludes by scalar implicature that four boys left, or that three boys and a girl left. (47.b) does not exclude that, because the alternatives are used by *at least* and then eliminated.

(49) a′. $[\![[[\varnothing\ \textit{three bóys}]_{F1}\ [t_1\ \textit{left}]]]\!] = \exists x[3(x) \wedge \text{BOYS}(x) \wedge \text{LEFT}(x)]$
$[\![[[\varnothing\ \textit{three bóys}]_{F1}\ [t_1\ \textit{left}]]]\!]_A = \{\langle \exists x[P(x) \wedge \text{LEFT}(x)], \exists x[Q(x) \wedge \text{LEFT}(x)]\rangle \mid P \leq_I Q\}$

b′. $[\![[[\varnothing\ \textit{at least}\ [\textit{three bóys}]_F]_1\ [t_1\ \textit{left}]]]\!]$
$= \exists x \exists Q[\lambda y[3(y) \wedge \text{BOYS}(x)] \leq_i Q \wedge Q(y) \wedge \text{LEFT}(y)]$
$[\![[[\varnothing\ \textit{at least}\ [\textit{three bóys}]_F]_1\ [t_1\ \textit{left}]]]\!]_A$ = the standard alternatives

In our next example, the ordering relation is introduced by a taxonomic hierarchy. Assume that we are interested in the origin of John. We know that he is an American, we even have evidence that he is a Texan, but we don't quite know yet his origin from within Texas. In this situation one can say:

(50) a. John is at least a Téxan$_F$.

We get this interpretation if *a Texan* is a node of a taxonomic relation $\leq_{TAX}$ on which we have, for example, $\text{AMERICAN} \leq_{TAX} \text{TEXAN}$, and $\text{TEXAN} \leq_{TAX} \text{AUSTINITE}$. We will get the following meaning:

(50) b. $\exists Q[\text{TEXAN} \leq_{TAX} Q \wedge Q(\text{JOHN})]$

That is, it is claimed that John has a property Q that is at least as specific as TEXAN on the taxonomic scale. Due to the nature of taxonomic scales this will entail that John is a Texan, and also that John is an American.

Finally, let us consider an example in which the order of the alternatives is introduced by a hierarchical relation.

(51) a. Mary is at least [an assóciate professor]$_F$.

The expression in focus, *an associate professor*, introduces the following ordered alternatives:

(51) b. $[\![[\textit{an associate professor}]_F]\!]_A$
$= \{\langle \text{ASSIST.PROF}, \text{ASSIST.PROF}\rangle, \langle \text{ASSIST.PROF}, \text{ASSOC.PROF}\rangle, \langle \text{ASSIST.PROF}, \text{FULLPROF}\rangle,$
$\langle \text{ASSOC.PROF}, \text{ASSOC.PROF}\rangle, \langle \text{ASSOC.PROF}, \text{FULLPROF}\rangle,$
$\langle \text{FULLPROF}, \text{FULLPROF}\rangle\}$
$= \leq_{PROF}$

This leads to the following meaning, which entails that Mary is either an associate professor or a full professor.

(51) c. $\exists Q[\text{ASSOC.PROF} \leq_{PROF} Q \wedge Q(\text{MARY})]$

Notice that in this case, a sentence of the form [...*at least* ...] does not entail the same sentence without *at least*. (51.a) does not entail that Mary is an associate professor; she might be a full professor.

3.4. *At most* Quantifiers

It seems straightforward to apply the account for *at least* developed in the previous section to downward-entailing quantifiers like *at most three boys*. The meaning that suggests itself for *at most* is the following (compare it with the meaning of *at least* given in (41)):

(52) $[\![at\ most\ \alpha]\!] = \bigcup \{X \mid \langle X, [\![\alpha]\!]\rangle \in [\![\alpha]\!]_A\}$
$[\![at\ most\ \alpha]\!]_A$ = the standard alternatives

That is, *at most* α creates as a meaning the union of all alternative meanings X of α that are at most as strong as the meaning of α. In the case of *at most thrée*$_F$ *boys*, these are the alternatives that are denoted by *three boys*, *two boys* and *one boy*.

(53) a. $[\![thrée_F\ boys]\!] = \lambda x[3(x) \wedge \text{BOYS}(x)]$
$[\![thrée_F\ boys]\!]_A = \{\langle \lambda x[n(x) \wedge \text{BOYS}(x)], \lambda x[m(x) \wedge \text{BOYS}(x)]\rangle \mid n \leq_N m\}$

b. $[\![at\ most\ [thrée_F\ boys]]\!] = \lambda x[\leq 3(x) \wedge \text{BOYS}(x)]$
$[\![at\ most\ [thrée_F\ boys]]\!]_A$ = the standard alternatives

We then get the meaning (54.b) for the sentence (54.a). If this sentence is asserted, it is claimed that there was an individual x consisting of at most three boys such that x left.

(54) a. At most thrée$_F$ boys left.

b. $[\![[[\varnothing\ [at\ most\ [thrée_F\ boys]]]_1\ [t_1\ left]]]\!] = \exists x[\leq 3(x) \wedge \text{BOYS}(x) \wedge \text{LEFT}(x)]$

There are at least two problems with this. One concerns the proposed meaning: It entails that there was at least one boy that left. But the following sentence does not express a contradiction:

(55) The evidence we have shows that at most thrée$_F$ boys left, but we are not sure whether any boy left.

The GQ analysis, $\#(\text{BOY} \cap \text{LEFT}) \leq 3$, fares better in this respect, as it does not entail that a boy left (notice that if the intersection is empty, we have $\#(\text{BOY} \cap \text{LEFT}) = 0$, and $0 \leq 3$).

The second problem is even more severe. It concerns the fact that (54.b) does not exclude that more than three boys left. As we have seen, cumulativity of predicates like *left* will ensure that, if four boys left, then the proposition 'three boys left' is true as well.

Perhaps we can exclude propositions like 'four boys left' using alternatives. For example, we might assume a rule for *at most* that keeps the alternatives alive:

(56) $[\![at\ most\ \alpha]\!] = \bigcup \{X \mid \langle X, [\![\alpha]\!]\rangle \in [\![\alpha]\!]_A\}$
$[\![at\ most\ \alpha]\!]_A = [\![\alpha]\!]_A$

The ASSERT operator then could state that there are reasons for the speaker not to claim the alternatives that are not entailed by the meaning. For our example, this amounts to saying that there are reasons not to claim propositions of the form 'n boys left', for $n > 3$. Hence, these propositions would be excluded by scalar implicature, just as with the sentence *three boys left* in the theory introduced in section (3.1).

But *three boys left* and *at most thrée*$_F$ *boys left* differ in their semantic behavior. The second sentence expresses a stronger commitment to the exclusion of alternatives. It does not just indicate that the speaker has reasons not to claim these alternative propositions, but more specifically says that these propositions are false. This shows up when we try to cancel the additional meaning component. (57.b) is a contradiction, in contrast to (57.a).

(57) a. Three boys left, perhaps even fóur$_F$.

b. *At most thrée$_F$ boys left, perhaps even fóur$_F$.

How can we express the different status of the alternatives in these cases? Notice that the alternatives were introduced in different ways. In (57.a), they are introduced by the meaning of number words (they belong to a Horn scale). In (57.b), we find that in addition focus is involved. We may now distinguish between alternatives that are just introduced by Horn scales, and alternatives that are introduced by focus. In the latter case, alternatives that are not entailed by the meaning proper are not only explicitly not asserted, but explicitly denied. This suggests the following rule for the ASSERT operator:

(58) ASSERT(M, A) =

a. If the alternatives were introduced by focus:
Speaker claims M, and for every $M' \in A$ such that $M \not\Rightarrow M'$, Speaker claims $\neg M'$.

b. Otherwise, Speaker claims M,
and, for every $M' \in A$, $M' \neq M$, speaker has reasons not to claim M'.

There may be some evidence for the particular role of focus in introducing alternatives. Van Kuppevelt (1996) observes that if a number word is the focus of an answer (in his terms, is part of the comment), then we have an "exact" interpretation. This explains the following contrast:

(59) a. [Who has fourteen children?]
Nígel$_F$ has fourteen children. In fact, he has fífteen$_F$.

b. [How many children does Nigel have?]
*Nigel has fóurteen$_F$ children. In fact, he has fífteen$_F$.

However, we find that (60) is fine, in contrast to (59.b).

(60) [How many children does Nigel have?]
Nigel has fóurteen$_F$ children, perhaps even fifteen$_F$.

It seems that, contrary to van Kuppevelt, focusation does not necessarily lead to an "exactly" reading. Under such an interpretation, (60) would be contradictory, just as **Nigel has exactly fourteen children, perhaps even fifteen*. The only way to make sense of this is to assume that (60) excludes by scalar implicature, and not by meaning, that Nigel has more children. The contrast observed in (59) has to be explained in different ways. In (59.a), the first clause is motivated in its context because it simply takes over the background of the question, *have fourteen children*, which indicates maximal coherence of the answer to the question. (An answer like *Nigel has fifteen children* is an indirect answer from which the direct answer can be inferred). In (59.b) there is no reason to give the answer *Nigel has fourteen children* in the first place, and therefore the interchange is bad.

Why, then, is (57.b) bad, or a sentence like **Nigel has at most fóurteen*$_F$ *children, perhaps even fífteen*$_F$? The obvious reason is that the first clause already excludes explicitly that Nigel has more than fourteen children. The second clause then expresses a blatant contradiction to the first clause.

The use of alternatives to capture the meaning of *at most* is also implausible because of the following reason. Clearly, the meanings of *at least* and *at most* should be related. But we have seen in the preceding section that we should assume that *at least* makes use of the alternatives introduced by focus, leaving no alternatives that could trigger scalar implicatures. Similarly, we should expect that *at most* makes use of the alternatives and does not leave any that could trigger scalar implicature.

One way of dealing with the special meaning contribution of *at most* is the following. We assume that semantic interpretations do not just give us truth conditions, but also "falsity conditions". That is, the interpretation of an expression is a pair that specifies the truth-conditional meaning and the falsity-conditional meaning. This suggestion has been made for semantic interpretation within Situation Semantics, cf. e.g. Barwise (1987). It can be used to characterize the meaning of particles like *at most* by assuming that this particle does not have any effect on the truth conditions, but rather affects the falsity conditions. A sentence like *at most thrée*$_F$ *boys left* says that the proposition 'more than three boys left' is false, and it leaves the truth conditions unspecified.

We might be able to develop an account along these lines. However, it has to face some fundamental problems, and it is unclear whether a principled solution can be given. Going from mere truth conditions to truth/falsity conditions enforces a fundamental change in the semantic representation, because now every expression should be characterized both in truth conditions and in falsity conditions. But then it is unclear what the *truth* conditions of a sentence like *at most thrée*$_F$ *boys came* should be. We could give a straightforward compositional analysis along the following lines: The sentence has as falsity condition $\exists x[>n(x) \wedge \textsc{boys}(x) \wedge \textsc{left}(x)]$, that is, this sentence is false, and as truth condition $\exists x[\leq 3(x) \wedge \textsc{boys}(x) \wedge \textsc{left}(x)]$. However, notice that this would wrongly entail that at least one boy left.

I would rather like to propose that *at most*-sentences do not have any meaning proper, but rather come with a type of alternatives that are marked as being excluded by the speaker. Let us assume that alternatives can have a **polarity**, a value that is either positive, marked "+", negative, marked "–", or neutral. Technically, positive and negative alternatives are pairs of a polarity marker and a meaning, but instead of $\langle +, a\rangle$ and $\langle –, a\rangle$ I will simply write +a and –a, respectively.

The rule for *at most* now can be given as follows. I use the notation –A to indicate that the set of alternatives A are all assigned the polarity marker "–", and similarly +A.

(61) $\llbracket \textit{at most } \alpha\rrbracket$ = undefined
$\llbracket \textit{at most } \alpha\rrbracket_A = -\{X \in \text{Field}(\llbracket\alpha\rrbracket_A) \mid \neg\langle X, \llbracket\alpha\rrbracket\rangle \in \llbracket\alpha\rrbracket_A\}$

That is, we reduce the set of alternatives of α to all those that are not smaller or equal to the meaning of α, and we mark these alternatives as negative.

Let me illustrate the derivation of example (54.a) in this new framework.

(62) a. $\llbracket \textit{thrée}_F \textit{ boys}\rrbracket = \lambda x[3(x) \wedge \textsc{boys}(x)]$
$\llbracket \textit{thrée}_F \textit{ boys}\rrbracket_A = \{\langle \lambda x[n(x) \wedge \textsc{boys}(x)], \lambda x[m(x) \wedge \textsc{boys}] \mid n \leq_N m\}$

b. $\llbracket \textit{at most thrée}_F \textit{ boys}\rrbracket$: undefined
$\llbracket \textit{at most thrée}_F \textit{ boys}\rrbracket_A$
$= -\{X \in \text{Field}(\llbracket \textit{thrée}_F \textit{ boys}\rrbracket_A) \mid \neg\langle X, \llbracket \textit{thrée}_F \textit{ boys}\rrbracket\rangle \in \llbracket \textit{thrée}_F \textit{ boys}\rrbracket_A\}$
$= -\{\lambda x[n(x) \wedge \textsc{boys}(x)] \mid 3 <_N n\}$
$= \{-\lambda x[n(x) \wedge \textsc{boys}(x)] \mid 3 <_N n\}$

c. $\llbracket \varnothing_{DET}\rrbracket = \lambda Q\lambda P\exists x[Q(x) \wedge P(x)]$
$\llbracket \varnothing_{DET}\rrbracket$ = the standard alternatives

d. $\llbracket[\varnothing \textit{ at most thrée}_F \textit{ boys}]\rrbracket$ = undefined
$\llbracket[\varnothing \textit{ at most thrée}_F \textit{ boys}]\rrbracket_A = \{-\lambda P\exists x[n(x) \wedge \textsc{boys}(x) \wedge P(x)] \mid 3 <_N n\}$

e. $\llbracket[t_1 \textit{ left}]\rrbracket = \textsc{left}(x_1)$
$\llbracket[t_1 \textit{ left}]\rrbracket_A = \{\textsc{left}(x_1)\}$

f. $\llbracket[[\varnothing \textit{ at most thrée}_F \textit{ boys}]_1 [t_1 \textit{ left}]]\rrbracket$ = undefined
$\llbracket[[\varnothing \textit{ at most thrée}_F \textit{ boys}]_1 [t_1 \textit{ left}]]\rrbracket = \{-\exists x[n(x) \wedge \textsc{boys}(x) \wedge \textsc{left}(x)] \mid 3 <_N n\}$

The definition of the ASSERT operator now contains in addition the following clause:

(63) $\textsc{assert}(M, A)$: If M is undefined, then for every $-M' \in A$: Speaker claims $\neg M'$

In the case at hand this amounts to the following:

(62) g. For all $M' \in \{-\exists x[n(x) \wedge \textsc{boys}(x) \wedge \textsc{left}(x)] \mid 3 <_N n\}$, Speaker claims $\neg M$,
that is, Speaker claims $\neg\exists x[>3(x) \wedge \textsc{boys}(x) \wedge \textsc{left}(x)]$

That is, Speaker claims that there is no sum individual consisting of more than three boys that left. No scalar implicature arises in this case, as we do not compare any meaning that is actually expressed with alternative meanings.

We have argued above that *at most* does not lead to existential presuppositions. However, in many cases sentences with *at most* seem to be close to having such presuppositions. We may express this by saying that, in addition to the meaning rule in (61) we have the following meaning rule:

(64) $\llbracket \textit{at most } \alpha\rrbracket: \bigcup \{X \mid \langle X, \llbracket\alpha\rrbracket\rangle \in \llbracket\alpha\rrbracket_A\}$
$\llbracket \textit{at most } \alpha\rrbracket_A = -\{X \mid \langle\llbracket\alpha\rrbracket, X\rangle \in \llbracket\alpha\rrbracket_A \wedge X \neq \llbracket\alpha\rrbracket\}$

Our example *at most thrée*$_F$ *boys left* would then have the meaning '$\exists x[\leq 3(x) \wedge \textsc{boys}(x) \wedge \textsc{left}(x)]$', which entails that (at least) one boy left. Notice that no scalar implicature will arise under this analysis either, because this meaning does not stand in any relation of semantic strength to the alternative meanings, which are of the form $\neg\exists x[n(x) \wedge \textsc{boys}(x) \wedge \textsc{left}(x)]$, with $n > 3$.

One piece of evidence that *at most* can be understood as having an existential presupposition comes from the fact that *at most* NPs can introduce discourse referents. However, discourse referents introduced by *at most* are not quite as good as NPs based on *at least*, and it might be that the pronoun in these cases is shorthand for definite descriptions (here, *the boys that left*) which can be accommodated if their presupposition is not satisfied.

(65) a. $^?$At most three boys left. They found the play boring.

b. At least three boys left. They found the play boring.

The introduction of polarity-marked alternatives may appear as heavy machinery for cases that GQ theory treated in a rather elegant way all along. This is certainly true when we just consider examples like *at most thrée*$_F$ *boys left*. But it gives us the proper representation in other cases, in which the GQ representation failed. It gives us the proper meanings in cases in which we have more than one *at most*-NPs. Consider the following example (I just specify the alternatives here):

(66) a. At most thrée$_F$ boys ate at most séven$_F$ apples.

b. $[\![[[\varnothing$ *at most séven*$_F$ *apples*$]_2$ $[t_1$ *ate* $t_2]]]\!]_A$
$= \{-\exists x[n(x) \wedge \textsc{apples}(x) \wedge \textsc{ate}(x)(x_1)] \mid n > 7\}$

c. $[\![[\varnothing$ *at most thrée*$_F$ *boys*$]]\!]_A$
$= \{-\lambda X\exists x[n(x) \wedge \textsc{boys}(x) \wedge X(x)] \mid n > 3\}$

d. $[\![[[\varnothing$ *at most thrée*$_F$ *boys*$]_1$ $[[\varnothing$ *at most séven*$_F$ *apples*$]_2$ $[t_1$ *ate* $t_2]]]]\!]_A$
$= \{-\exists y\exists x[n(x) \wedge \textsc{boys}(x) \wedge m(y) \wedge \textsc{apples}(y) \wedge \textsc{ate}(y)(x)] \mid n > 3, m > 7\}$

I assume here that the combination of two negative alternatives will lead to a negative alternative, following the rule

(67) $f(-X, -Y) = -f(X, Y)$

The assertion of (66) gives us the following:

(66) e. $\textsc{assert}((66))$:
For all $M \in \{-\exists y\exists x[n(x) \wedge \textsc{boys}(x) \wedge m(y) \wedge \textsc{apples}(y) \wedge \textsc{ate}(y)(x)] \mid n{>}3 \text{ or } m{>}7\}$,
Speaker claims $\neg M$,
that is, Speaker claims $\neg\exists y\exists x[>3(x) \wedge \textsc{boys}(x) \wedge \textsc{apples}(y) \wedge \textsc{ate}(y)(x)]$, and
$\neg\exists y\exists x[\textsc{boys}(x) \wedge >5(y) \wedge \textsc{apples}(y) \wedge \textsc{ate}(y)(x)]$

These are the correct truth conditions: It is denied that more than three boys ate apples, and that more than seven apples were eaten by boys.

Let us turn to cases in which the focus determines a different type of alternatives, as in the following example. Recall that the relation $\leq_I$ is a relation for sets of individuals, based on the part relation on individuals, $\leq_i$ (cf. (48)).

(68) a. At most [three bóys]$_F$ left.

b. $[\![$*at most* $[$*three bóys*$]_F]\!]$
$= -\{P \in \text{Field}([\![[\textit{three bóys}]_F]\!]_A) \mid \neg \langle P, [\![[\textit{three bóys}]_F]\!]\rangle \in [\![\textit{three bóys}]_F]\!]_A\}$
$= \{-P \mid \neg\, P \leq_I \lambda x[3(x) \wedge \textsc{boys}(x)]\}$

c. $[\![[[\varnothing$ *at most* $[$*three bóys*$]_F]_1$ $[t_1$ *left*$]]]\!]_A$
$= \{-\exists x[P(x) \wedge \textsc{left}(x)] \mid \neg\, P \leq_I \lambda x[3(x) \wedge \textsc{boys}(x)]\}$

This set contains all propositions of the form 'some P left', where P ranges over sets that are not an I-part of the set that contains sum individuals consisting of three boys. That is, we will exclude propositions like 'three boys left', 'two boys left', 'one boy left', and 'John left' (if John is a boy) from this set. But propositions like 'four boys left', or 'three boys and a girl left', or 'Sue left' (if Sue is not a boy) will be in this set. Asserting (68) amounts to denying these propositions. (I have assumed that the field of $<_I$ is the set of all subsets of the universe, and hence I have omitted this condition.)[6]

Finally, notice that we get the right representation for a sentence based on a hierarchical relation:

(69) a. Mary is at most [an assóciate professor]$_F$

b. $[\![$*Mary is at most* [*an assóciate professor*]$_F]\!]_A$
$= \{-X(\text{MARY}) \mid X \in \text{Field}(\leq_{PROF}) \wedge \neg \langle X, \text{ASSOC.PROF}\rangle\}$

The assertion of this sentence will lead to the claim that Mary does not have a higher rank than the one of an associate professor.

3.5. Other Particles and Quantifiers

In the preceding section I have introduced a semantic theory for the particle *at most*. In this section I will show how it can be extended to handle other particles and quantifiers.

First, if we want to give a parallel treatment to *at least* and *at most*, then we might replace the meaning rule for *at least* given in (41) by the following rule:

(70) $[\![$*at least* $\alpha]\!]$ = undefined
$[\![$*at least* $\alpha]\!] = +\{X \mid \langle X, [\![\alpha]\!]\rangle \in [\![\alpha]\!]_A\}$

The ASSERT rule should then be extended as follows for positive alternatives:

(71) ASSERT(M, A): If M is undefined, and if A contains positively marked alternatives, then Speaker claims that there is one $+M' \in A$ such that M' is true.

Under this analysis, the sentence *at least thrée*$_F$ *boys left* has positively marked alternatives 'three boys left', 'two boys left', 'one boy left'. If asserted, the speaker is committed to the claim that there is one alternative that is true. Notice that these are the same truth conditions as before, but now expressed by way of positively marked alternatives. We can explain the slightly different status of *at least three boys* and *at most three boys* discussed in (65) as follows: If there is one alternative proposition that can be asserted entails that, when this proposition is in fact asserted, a discourse referent for the indefinite is introduced. We find a similar introduction of discourse referents in disjunctions, as in the following case:

(72) Pedro owns a donkey, or he owns a horse. In any case, he beats it / the animal.

The particle *more than* can be analyzed as follows; compare this with the meaning rule for *at least*.

(73) $[\![$*more than* $\alpha]\!]$ = undefined
$[\![$*more than* $\alpha]\!]_A = \cup \{X \mid \langle [\![\alpha]\!], X\rangle \in [\![\alpha]\!]_A \wedge X \neq [\![\alpha]\!]_A\}$

The particle *less than* will be analyzed as follows; compare with *at most* as defined in (61) and (68):

(74) $[\![$*less than* $\alpha]\!]$ = undefined
$[\![$*less than* $\alpha]\!]_A = -\{X \in \text{Field}([\![\alpha]\!]_A) \mid \neg\langle X, [\![\alpha]\!]\rangle \in [\![\alpha]\!]_A \vee X = [\![\alpha]\!]_A\}$

The two-place particle *between* can be analyzed as a combination of *at least* and *at most*:

[6] The reason why I did not define the alternatives as $-\{X \mid \langle [\![\alpha]\!], X\rangle \in [\![\alpha]\!]_A \wedge X \neq [\![\alpha]\!]\}$, but rather by the complement operation as in (61), is that under this definition, (68.c) would not exclude propositions like 'three girls left', or 'Sue left'.

(75) $[\![\textit{between } \alpha \textit{ and } \beta]\!]$ = undefined
$[\![\textit{between } \alpha \textit{ and } \beta]\!]_A$ = $+\{X \mid \langle X, [\![\alpha]\!]\rangle \in [\![\alpha]\!]_A\} \cup$
$-\{X \in \text{Field}([\![\beta]\!]_A \mid \neg \langle [\![\beta]\!], X\rangle \in [\![\beta]\!]_A\}$
$= [\![\textit{at least } \alpha]\!]_A \cup [\![\textit{at most } \beta]\!]_A$

(76) $[\![\textit{exactly } \alpha]\!]$ = undefined
$[\![\textit{exactly } \alpha]\!]_A$ = $+\{X \mid \langle X, [\![\alpha]\!]\rangle \in [\![\alpha]\!]_A\} \cup -\{X \in \text{Field}([\![\alpha]\!]_A \mid \neg \langle [\![\alpha]\!], X\rangle \in [\![\alpha]\!]_A\}$
$= [\![\textit{at least } \alpha]\!]_A \cup [\![\textit{at most } \alpha]\!]_A$

In this case the alternatives will end up containing both positive and negative elements. As an example, consider the interpretation of *between three and five boys ate between seven and ten apples*:

(77) a. $[\![\textit{thr\'ee}_F]\!] = \lambda P\lambda x[3(x) \wedge P(x)]$
$[\![\textit{thr\'ee}_F]\!]_A = \{\langle \lambda P\lambda x[n(x) \wedge P(x)], \lambda P\lambda x[m(x) \wedge P(x)]\rangle \mid n <_N m\}$

b. $[\![\textit{f\'ive}_F]\!] = \lambda P\lambda x[5(x) \wedge P(x)]$
$[\![\textit{f\'ive}_F]\!]_A = \{\langle \lambda P\lambda x[n(x) \wedge P(x)], \lambda P\lambda x[m(x) \wedge P(x)]\rangle \mid n <_N m\}$

c. $[\![\textit{between thr\'ee}_F \textit{ and f\'ive}]\!]_A$
$= +\{X \mid \langle X, [\![\textit{thr\'ee}_F]\!]\rangle \in [\![\textit{thr\'ee}_F]\!]_A\} \cup -\{X \in \text{Field}([\![\textit{f\'ive}]\!]_A \mid \neg \langle [\![\textit{f\'ive}]\!], X\rangle \in [\![\textit{f\'ive}]\!]_A\}$
$= \{+\lambda P\lambda x[n(x) \wedge P(x)] \mid 3 \leq_N n\} \cup \{-\lambda P\lambda x[n(x) \wedge P(x)] \mid 5 <_N n\}$

d. $[\![\textit{between thr\'ee}_F \textit{ and f\'ive}_F \textit{ boys}]\!]_A$
$= \{+\lambda x[n(x) \wedge \text{BOYS}(x)] \mid 3 \leq_N n\} \cup \{-\lambda x[n(x) \wedge \text{BOYS}(x)] \mid 5 <_N n\}$

e. $[\![\varnothing \textit{ between thr\'ee}_F \textit{ and f\'ive}_F \textit{ boys}]\!]_A$
$= \{+\lambda P\exists x[n(x) \wedge \text{BOYS}(x) \wedge P(x)] \mid 3 \leq_N n\} \cup \{-\lambda P\exists x[n(x) \wedge \text{BOYS}(x) \wedge P(x)] \mid 5 <_N n\}$

f. $[\![\varnothing \textit{ between s\'even}_F \textit{ and t\'en}_F \textit{ apples}]\!]_A$
$= \{+\lambda P\exists x[n(x) \wedge \text{APPLES}(x) \wedge P(x)] \mid 7 \leq_N n\}$
$\cup \{-\lambda P\exists x[n(x) \wedge \text{APPLES}(x) \wedge P(x)] \mid 10 <_N n\}$

g. $[\![[\varnothing \textit{ between s\'even}_F \textit{ and t\'en}_F \textit{ apples}]_2 [t_1 \textit{ ate } t_2]]\!]_A$
$= \{+\exists x[n(x) \wedge \text{APPLES}(x) \wedge \text{ATE}(x_1, x)] \mid 7 \leq_N n\}$
$\cup \{-\exists x[n(x) \wedge \text{APPLES}(x) \wedge \text{ATE}(x_1, x)] \mid 10 <_N n\}$

h. $[\![[[\varnothing \textit{ between thr\'ee}_F \textit{ and f\'ive}_F \textit{ boys}]_1 [[\varnothing \textit{ between s\'even}_F \textit{ and t\'en}_F \textit{ apples}]_2 [t_1 \textit{ ate } t_2]]]\!]_A$
$= \{+\exists x\exists y[n(x) \wedge \text{BOYS}(x) \wedge m(y) \wedge \text{APPLES}(y) \wedge \text{ATE}(x, y)] \mid 3 \leq_N n \text{ and } 7 \leq_N m\}$
$\cup \{-\exists x\exists y[n(x) \wedge \text{BOYS}(x) \wedge m(y) \wedge \text{APPLES}(y) \wedge \text{ATE}(x, y)] \mid 5 <_N n \text{ or } 10 <_N m\}$

We get this result under the following assumption about how positive and negative alternatives should be combined, which says that whenever a negative alternative is combined with any other alternative, the result will be negative. A positive alternative can be obtained only by combining positive alternatives.

(78) $f(-X, -Y) = f(-X, +Y) = f(+X, -Y) = -f(X, Y);$
$f(+X, +Y) = +f(X, Y)$

If asserted, (77) amounts to the following: The speaker claims that there is a proposition of the form 'n boys ate m apples', with n ≤ 3 and m ≤ 7, that is true, but all propositions of the form 'n boys ate m apples', where either 5 < n or 10 < m, is false.

We also have to account for the combination of negative and positive alternatives with neutral alternatives, as in the following sentences:

(79) a. At least three boys ate seven apples.

b. At most three boys ate seven apples.

In the cumulative interpretation, (79.a,b) say that seven apples were eaten by boys, and in addition, (a) three or more boys participated in the eating and (b) it is not the case that more than three boys participated in the eating. We get these interpretations by assuming that non-neutral alternatives take precedence over neutral ones:

(80) $f(X, \pm Y) = f(\pm X, Y) = \pm f(X, Y)$

One interesting consequence of this treatment of expressions like *at most*, *at least*, *exactly*, and *between n and m* is that on the semantic level they are essentially indefinites. There is one well-known property that they share with other indefinites, namely, they occur in *there*-sentences, in contrast to true quantifiers.

(81) a. There were three / at least three / at most three / exactly three / between three and five books on the table.

b. *There was the book / every book on the table.

c. *There were most books on the table.

There exist a number of attempts to characterize this distribution semantically. A recent one, McNally (to appear), essentially assumes that *there*-sentences have predicates as subjects (type $\langle e, t\rangle$). This excludes true quantifiers from this position. Notice that the theory developed here analyzes expressions like *three books*, *at most three books*, or *between three and five books* as predicates, and hence predicts that we find these expressions as the subjects of *there*-sentences.

3.6. The Translation of Alternatives to Meanings

Let me turn now to an important general point, and a final twist in our story. One feature of the theory developed here is that expressions like *at least*, *at most*, *exactly* etc. are interpreted in an two-pronged way. The particles themselves, together with the alternatives introduced by a focus within their scope, create a certain type of alternative set. Another operator, an illocutionary operator like ASSERT, makes use of this information and transforms it into something that can be related to the usual truth conditions that we ascribe to these sentences. For example, the alternatives in (62.f) were transformed by ASSERT to the formula $\neg\exists x[>3(x) \wedge \text{BOYS}(x) \wedge \text{LEFT}(x)]$.

So far we have assumed that illocutionary operators perform that transformation, and as we have seen that these particles can block scalar implicatures that presumably arise by illocutionary operations, this appears quite plausible. However, the particles in question can be interpreted in terms of truth conditions even when no illocutionary operator is present. Consider the following example with the particle *at most*:

(82) Mary was aware that at most thrée boys were present.

The prominent reading of (82) is one in which can be described as 'Mary was aware that the number of boys that were present is at most three'. We need the truth-conditional interpretation within the description of Mary's belief, even though the embedded clause is not directly asserted. One way of dealing with that is to assume the creation of truth conditions from a set of alternatives can be enacted without the presence of an illocutionary operator. As we need propositions to express that rule, it is plausible to assume that, as soon as we arrive in the semantic derivation at the propositional level (type t), the marked alternatives are translated to truth conditions.

(83) If α is of type t, $[\![\alpha]\!]$ is undefined, and $[\![\alpha]\!]_A$ contains ±-marked alternatives,
then take as new meaning $[\![\alpha]\!]$ the following: $\bigcup\, [\{p \mid +p \in [\![\alpha]\!]_A\} \cup \{\neg p \mid -p \in [\![\alpha]\!]_A\}]$,
and as the new alternatives $[\![\alpha]\!]_A$ the standard alternatives (that is, $\{[\![\alpha]\!], \langle[\![\alpha]\!], [\![\alpha]\!]\rangle$).

This rule is now independent of any illocutionary operator, and hence it allows us to deal with cases like (82). It gives us the right result in root sentences like *At most thrée*$_F$ *boys were present* as well: they express a proposition, here 'it is not the case that more than three boys were present'. The ASSERT operator is applied to this proposition, and as it does not have proper alternatives, no scalar implicature will arise.

The point at which marked alternatives are translated into meanings cannot be identified as the syntactic category S. Consider the following example (from the British National Corpus):

(84) In 1621 the French mathematician Bachet de Meziriac observed that apparently every positive number could be expressed as a sum of at most four squares.

The embedded clause certainly is not intended to mean: 'The highest n such that every positive number can be expressed as a sum of n squares is 4'. Rather, it should express the following: 'Every positive number can be expressed by a sum $x_1^2 + x_2^2 + ... x_n^2$, where $x_1^2 + x_2^2 + ... x_n^2$ has the property that $n \leq 4$'. The critical point is that the last NP, *at most four squares*, introduces a referent, and the description of this referent is that it is a sum of squares that does not contain more than 4 elements. Hence the point at which the alternatives are translated to a truth-conditional meaning is within the application of the descriptive content of the NP, *at most four squares*, to its referent. This is not a clause in any syntactic sense, but a proposition (type t). in semantic interpretation. Hence the translation of alternatives to meanings follows semantic criteria, not syntactic ones.

We find similar types of interpretations for other particles. Consider the following contrast:

(85) a. He only can play the piano with his léft$_F$ hand.

b. He can play the piano with only his léft$_F$ hand.

Example (85.b) can be paraphrased as: 'He can play the piano with x, and x is only his left hand, that is, x does not contain his right hand.'

Finally, I would like to mention that the particles under consideration are all focus-sensitive, and they share with other focus-sensitive particles that both their scope and their focus matter for interpretation (for example, in (85.a,b), the particle *only* had the same focus, but different scope). We have seen the influence of focus above (cf. *At most thrée$_F$ boys left* vs. *At most* [*three bóys*]$_F$ *left*). The scope is indicated by the syntactic position of the particle. Contrast example (82) with the following:

(86) Mary was at most aware that thrée$_F$ boys were present.

This is to be paraphrased as: 'There is no n, n > 3, such that Mary was aware that n boys were present'.

I should add that it seems that not only ±-marked alternatives are subject to narrow-scope transformation to truth conditions. We also find this with normal number words that arguably introduce unmarked alternatives. Notice that the following sentence is not pragmatically odd:

(87) Three boys ate seven apples, and four boys ate nine apples.

We are talking here about two different groups of boys, and two different groups of apples. Scalar implicature applies on the level of the description of those groups, not on the level of the clause. The paraphrase of the first clause of (87) is 'There is an x and a y and x ate y, and x are three boys, and y are seven apples'. The subpredications 'x are three boys' and 'y are seven apples' are strengthened by scalar implicature to 'x are not more than three boys' and 'y are not more than seven apples'. What is important here is that we have two independent applications of scalar implicature. The second clause of (87) then introduces another group of boys and apples, with independent applications of scalar implicatures. Of course, there must be some criteria that allow us to distinguish between the two groups of boys.

4. Conclusion

In this article I tried to show that many expressions that were analyzed as determiners in Generalized Quantifier theory acquire their quantificational effect in a rather indirect way. They are particles that associate with an expression with focus that introduces alternatives, and they exploit these alternatives at different positions. This explains their syntactic distribution, which is considerably wider than the one of determiners. It also explains why meanings of sentences that contain them vary in the typical ways that we observe with other focus-sensitive particles. I tried to show why these particles do not trigger scalar implicatures that we normally find with number words. We had to work with the concept of ordered alternatives, and for the treatment of downward-entailing particles, like *at most*, we had to introduce the concept of positively marked and negatively marked alternatives. And we had to assume that alternatives can be translated into regular meanings at certain points in the syntactic derivation.

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